

ON THE GLOBAL 2-HOLONOMY FOR A 2-CONNECTION ON A 2-BUNDLE

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ABSTRACT. A crossed module constitutes a strict 2-groupoid \mathcal{G} and a \mathcal{G} -valued 2-cocycle on a manifold defines a 2-bundle. A 2-connection on this 2-bundle is given by a Lie algebra \mathfrak{g} valued 1-form A and a Lie algebra \mathfrak{h} valued 2-form B over each coordinate chart together with 2-gauge transformations between them, which satisfy the compatibility condition. Locally, the path-ordered integral of A gives us the local 1-holonomy, and the surface-ordered integral of (A, B) gives us the local 2-holonomy. The transformation of local 2-holonomies from one coordinate chart to another is provided by the transition 2-arrow, which is constructed from a 2-gauge transformation. We can use the transition 2-arrows and the 2-arrows provided by the \mathcal{G} -valued 2-cocycle to glue such local 2-holonomies together to get a global one, which is well defined.

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1. INTRODUCTION

Higher gauge theory is a generalization of gauge theory that describes the dynamics of higher dimensional extended objects. See e.g. [3] [4] [9] [17] for 2-gauge theory and [14] [18] [23] for 3-gauge theory. It involves higher algebraic structures and higher geometrical structures in

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mathematics: higher groups, higher bundles (gerbes) and higher connections, etc. (cf. e.g. [1] [5] [6] [7] [10] [15] and references therein). An important physical quantity in 2-gauge theory is the Wilson surface [8] [17]. This is a 2-dimensional generalization of Wilson loop or holonomy in differential geometry. We will discuss the global 2-holonomy for a 2-connection on a 2-bundle.

Let us recall definitions of 2-bundles and 2-connections. Suppose that $(G, H, \alpha, \triangleright)$ is a crossed module, where $\alpha : H \rightarrow G$ is a homomorphism of Lie groups and \triangleright is a smooth left action of G on H by automorphisms. Similarly, $(\mathfrak{g}, \mathfrak{h}, \alpha, \triangleright)$ is a differential crossed module, where $\alpha : \mathfrak{h} \rightarrow \mathfrak{g}$ is a homomorphism of Lie algebras and \triangleright is a smooth left action of \mathfrak{g} on \mathfrak{h} by automorphisms. A *local 2-connection* over an open set U is given by a \mathfrak{g} -valued 1-form A and a \mathfrak{h} -valued 2-form B over U such that

$$(1.1) \quad dA + A \wedge A = \alpha(B).$$

A *2-gauge-transformation* from a local 2-connection (A, B) to another one (A', B') is given by a G -valued function g and a \mathfrak{h} -valued 1-form φ such that

$$(1.2) \quad \begin{aligned} g \triangleright A' &= -\alpha(\varphi) + A + dg \cdot g^{-1}, \\ g \triangleright B' &= B - d\varphi - A \triangleright \varphi + \varphi \wedge \varphi. \end{aligned}$$

Given a crossed module $(G, H, \alpha, \triangleright)$, there exists an associated strict 2-groupoid denoted by \mathcal{G} . A *2-bundle* over a manifold M is given by a nonabelian \mathcal{G} -valued *2-cocycle* on M . This is a collection of (U_i, g_{ij}, f_{ijk}) , where $\{U_i\}_{i \in I}$ is an open cover of the manifold M , $g_{ij} : U_i \cap U_j \rightarrow G$ and $f_{ijk} : U_i \cap U_j \cap U_k \rightarrow H$ are smooth maps satisfying

$$(1.3) \quad \alpha\left(f_{ijk}^{-1}\right) g_{ij} g_{jk} = g_{ik},$$

and the *2-cocycle condition*

$$(1.4) \quad g_{ij} \triangleright f_{jkl} f_{ijl} = f_{ijk} f_{ikl}.$$

A *2-connection* on this 2-bundle over M is given by a collection of local 2-connections (A_i, B_i) over each coordinate chart U_i , together with a 2-gauge transformation (g_{ij}, a_{ij}) over each intersection $U_i \cap U_j$ from the local 2-connection (A_i, B_i) to another one (A_j, B_j) . They satisfy the following *compatibility condition*:

$$(1.5) \quad a_{ij} + g_{ij} \triangleright a_{jk} = f_{ijk} a_{ik} f_{ijk}^{-1} + A_i \triangleright f_{ijk} f_{ijk}^{-1} + df_{ijk} f_{ijk}^{-1},$$

over each triple intersection $U_i \cap U_j \cap U_k$. Note that minus signs in (1.2) become plus if φ is replaced by $-\varphi$. See also Remark 2.1 and 4.4 for this form of 2-gauge-transformations and the compatibility condition.

Given a \mathfrak{g} -valued 1-form A on an open set U , the 1-holonomy $F_A(\rho)$ along a Lipschitzian path $\rho : [a, b] \rightarrow U$ is well defined. It is given by the path-ordered integral. More precisely, $F_A(\rho)$ is the unique solution to the ODE

$$(1.6) \quad \frac{d}{dt} F_A(\rho_{[a,t]}) = F_A(\rho_{[a,t]}) \rho^* A_t \left(\frac{\partial}{\partial t} \right)$$

with the initial condition $F_A(\rho_{[a,t]})|_{t=a} = 1_G$, where $\rho_{[a,t]}$ is the restriction of the curve ρ to $[a, t]$ and $\rho^* A_t$ is the value of the pull back \mathfrak{g} -valued 1-form $\rho^* A$ at $t \in [a, b]$. Moreover, we can

integrate the 2-connection (A_i, B_i) along a surface $\gamma : [0, 1]^2 \rightarrow U_i$ to get a 2-arrow in \mathcal{G} , called the *local 2-holonomy*. It is a surface-ordered integral. If we denote the boundary of γ as follows

$$(1.7) \quad \begin{array}{ccc} y_1 & \xrightarrow{\gamma^u} & y_2 \\ \gamma^l \downarrow & & \downarrow \gamma^r \\ x_1 & \xrightarrow{\gamma^d} & x_2 \end{array} ,$$

the local 2-holonomy is a 2-arrow in \mathcal{G} :

$$(1.8) \quad \begin{array}{ccc} \bullet & \xrightarrow{F_{A_i}(\gamma^u)} & \bullet \\ F_{A_i}(\gamma^l) \downarrow & & \downarrow F_{A_i}(\gamma^r) \\ \bullet & \xrightarrow{F_{A_i}(\gamma^d)} & \bullet \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad \begin{array}{c} H_{A_i, B_i}(\gamma) \end{array} .$$

It was proved by Schreiber and Waldorf [20] that there exists a bijection between 2-connections on the trivial 2-bundle and 2-functors (play the role of 2-holonomy):

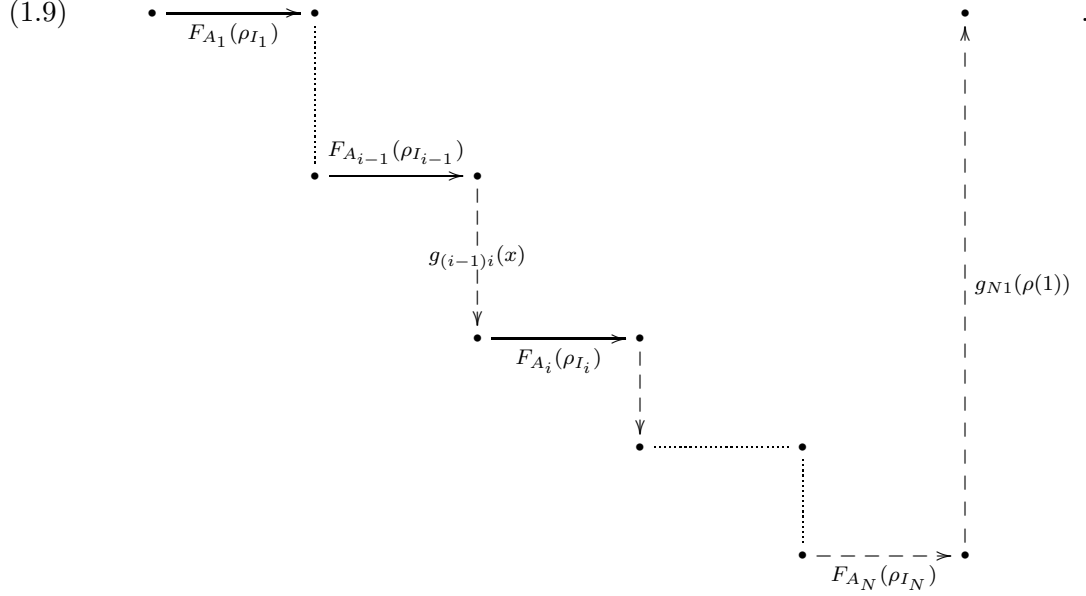
$$\{\text{smooth 2-functors } \mathcal{P}_2(M) \rightarrow \mathcal{G}\} \cong \{A \in \Lambda^1(M, \mathfrak{g}), B \in \Lambda^1(M, \mathfrak{h}); dA + A \wedge A = \alpha(B)\},$$

where $\mathcal{P}_2(M)$ is path 2-groupoid of manifold M . The local 1- and 2-holonomies are well defined. See also Martins-Picken [12] for the theory of local 1- and 2-holonomies. The problem is how to define the global 2-holonomy for a 2-connection on a nontrivial 2-bundle. This is known for Abelian 2-bundles by Mackaay-Picken [11]. Schreiber and Waldorf [21] proved the equivalence of several 2-categories associated a 2-connection to show the existence of the transport functor, which plays the role of global 2-holonomy. Parzygnat [16] studied its generalization, explicit computations and application to magnetic monopoles. On the other hand, Martins and Picken [13] introduced the notion of parallel transport by using the language of double groupoids. They also give the method of glueing local 2-holonomies to get a global one for the cubical version. This is a cubical description, rather than a simplicial description (see also Soncini-Zucchini [22] for this approach). A cubical 2-bundle does not seem to be a direct generalization of the ordinary case of (principal) bundles and connections. Recently Arias Abad and Schätz [2] compared these two approaches locally. In this paper we will give an elementary approach to this problem, including an algorithm to calculate the global 2-holonomy.

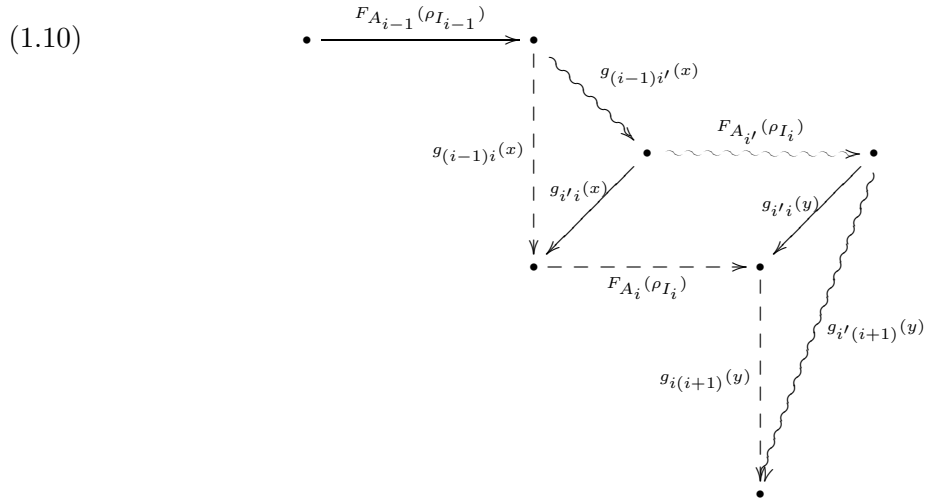
As a model, let us consider first how to glue local 1-holonomies to get a global one. Recall that a 1-connection on M is given by a collection of local 1-forms A_i over coordinate charts U_i , together with transition functions g_{ij} on $U_i \cap U_j$, which satisfy the 1-cocycle condition. They satisfy the following *compatibility condition*:

$$A_i = g_{ij}^{-1} A_j g_{ij} + g_{ij}^{-1} dg_{ij}$$

over $U_i \cap U_j$. Let $\rho : [0, 1] \rightarrow M$ be a loop, i.e., $\rho(0) = \rho(1)$. We divide the interval $[0, 1]$ into several subintervals $I_i := [t_i, t_{i+1}]$, $i = 1, \dots, N$, such that the image $\rho(I_i)$ is contained in a coordinate chart denoted by U_i . We have local 1-holonomies $F_{A_i}(\rho_{I_i})$. We glue $F_{A_{i-1}}(\rho_{I_{i-1}})$ with $F_{A_i}(\rho_{I_i})$ by the gauge transformation $g_{(i-1)i}(x)$ at point $x = \rho(t_i)$ to get the following path:



The composition of elements of G along this path is the global 1-holonomy of the connection along the loop ρ . Its conjugacy class is independent of the choice of the open sets U_i containing the paths $\rho(I_i)$. This is because that if we use $U_{i'}$ and $A_{i'}$ instead of U_i and A_i , respectively, we have the following commutative diagram



where $x = \rho(t_i)$, $y = \rho(t_{i+1})$. Here the 1-cocycle condition implies the commutativity of two triangles, and $g_{i'i}$ as a gauge transformation provides the commutative quadrilateral. The wavy path is what we obtain when U_i and A_i are replaced by $U_{i'}$ and $A_{i'}$, respectively. So the 1-arrows represented by the wavy and dotted paths coincide. When $U_1 = U_N$ is replaced by $U_{1'}$, we get the conjugacy of the global 1-holonomy by the element $g_{1'1}(\rho(0))$.

To construct the the global 2-holonomy, we consider a surface given by the union of the mapping γ in (1.7) and a mapping $\tilde{\gamma} : [0, 1]^2 \longrightarrow U_j$,

$$\begin{array}{ccc} y_2 & \xrightarrow{\tilde{\gamma}^u} & y_3 \\ \tilde{\gamma}^l \downarrow & & \downarrow \tilde{\gamma}^r \\ x_2 & \xrightarrow{\tilde{\gamma}^d} & x_3 \end{array} ,$$

such that the left path $\tilde{\gamma}^l$ above coincides with the right path γ^r in (1.7). Then we also have the following 2-arrow

$$(1.11) \quad \begin{array}{ccc} \bullet & \xrightarrow{F_{A_j}(\tilde{\gamma}^u)} & \bullet \\ \downarrow F_{A_j}(\tilde{\gamma}^l) & \searrow H_{A_j, B_j}(\tilde{\gamma}) & \downarrow F_{A_j}(\tilde{\gamma}^r) \\ \bullet & \xrightarrow{F_{A_j}(\tilde{\gamma}^d)} & \bullet \end{array}$$

in \mathcal{G} by the surface-ordered integration of the 2-connection (A_j, B_j) over U_j . The path $\tilde{\gamma}^l$ coincides with γ^r , but the local 1-holonomy $F_{A_j}(\tilde{\gamma}^l)$ in (1.11) is usually different from $F_{A_i}(\gamma^r)$ in (1.8). So we can not glue the 2-arrows in (1.8) and (1.11) directly. But we can integrate the 2-gauge transformation (g_{ij}, a_{ij}) along the path $\rho = \gamma^r = \tilde{\gamma}^l$ in $U_i \cap U_j$ to get the 2-arrow

$$(1.12) \quad \begin{array}{ccc} \bullet & \xrightarrow{F_{A_i}(\rho)} & \bullet \\ \downarrow g_{ij}(y_2) & \searrow \psi_{ij}(\rho) & \downarrow g_{ij}(x_2) \\ \bullet & \xrightarrow{F_{A_j}(\rho)} & \bullet \end{array}$$

in the 2-groupoid \mathcal{G} . We call this 2-arrow (1.12) the *transition 2-arrow along the path ρ* . It can be used to connect two arrows (1.8) and (1.11) to get

$$\begin{array}{ccccc} \bullet & \xrightarrow{F_{A_i}(\gamma^u)} & \bullet & \xrightarrow{g_{ij}(y_2)} & \bullet & \xrightarrow{F_{A_j}(\tilde{\gamma}^u)} & \bullet \\ \downarrow F_{A_i}(\gamma^l) & \searrow H_{A_i, B_i}(\gamma) & \downarrow F_{A_i}(\gamma^r) & \searrow \psi_{ij}^{-1}(\gamma^r) & \downarrow F_{A_j}(\tilde{\gamma}^l) & \searrow H_{A_j, B_j}(\tilde{\gamma}) & \downarrow F_{A_j}(\tilde{\gamma}^r) \\ \bullet & \xrightarrow{F_{A_i}(\gamma^d)} & \bullet & \xrightarrow{g_{ij}(x_2)} & \bullet & \xrightarrow{F_{A_j}(\tilde{\gamma}^d)} & \bullet \end{array} .$$

Now consider 4 adjacent rectangles $\gamma^{(\alpha)} : [0, 1]^2 \rightarrow U_\alpha$, $\alpha = i, j, k, l$,

$$(1.13) \quad \begin{array}{ccccc} z_1 & \xrightarrow{\quad} & z_2 & \xrightarrow{\quad} & z_3 \\ \downarrow & & \downarrow & & \downarrow \\ & \gamma^{(l)} & & \gamma^{(k)} & \\ \downarrow & & \downarrow & & \downarrow \\ y_1 & \xrightarrow{\quad} & y_2 & \xrightarrow{\quad} & y_3 \\ \downarrow & & \downarrow & & \downarrow \\ & \gamma^{(i)} & & \gamma^{(j)} & \\ \downarrow & & \downarrow & & \downarrow \\ x_1 & \xrightarrow{\quad} & x_2 & \xrightarrow{\quad} & x_3 \end{array}$$

in four different coordinate charts. We can connect the local 2-holonomies by using the transition 2-arrows along their common boundaries to get the following diagram:

$$(1.14) \quad \begin{array}{ccccc} \bullet & \xrightarrow{FA_l} & \bullet & \xrightarrow{g_{lk}(z_2)} & \bullet & \xrightarrow{FA_k} & \bullet \\ \downarrow FA_l & \nearrow H_{A_l, B_l}(\gamma^{(l)}) & \downarrow FA_l & \nearrow \psi_{lk}^{-1} & \downarrow FA_k & \nearrow H_{A_k, B_k}(\gamma^{(k)}) & \downarrow FA_k \\ \bullet & \xrightarrow{FA_l} & \bullet & \xrightarrow{g_{lk}(y_2)} & \bullet & \xrightarrow{FA_k} & \bullet \\ \downarrow g_{li}(y_1) & \nearrow \psi_{li} & \downarrow g_{li}(y_2) & \nearrow g_{kj}(y_2) & \downarrow g_{kj}(y_3) & \nearrow \psi_{kj} & \downarrow g_{kj}(y_3) \\ \bullet & \xrightarrow{FA_i} & \bullet & \xrightarrow{g_{ij}(y_2)} & \bullet & \xrightarrow{FA_j} & \bullet \\ \downarrow FA_i & \nearrow H_{A_i, B_i}(\gamma^{(i)}) & \downarrow FA_i & \nearrow \psi_{ij}^{-1} & \downarrow FA_j & \nearrow H_{A_j, B_j}(\gamma^{(j)}) & \downarrow FA_j \\ \bullet & \xrightarrow{FA_i} & \bullet & \xrightarrow{g_{ij}(x_2)} & \bullet & \xrightarrow{FA_j} & \bullet \end{array}$$

We add the following 2-arrow in \mathcal{G} in the central rectangle:

$$(1.15) \quad \begin{array}{ccccc} & & g_{lk}(y_2) & & \\ & & \nearrow f_{lkj}(y_2) & & \\ g_{li}(y_2) & & \searrow g_{lj}(y_2) & & g_{kj}(y_2) \\ & & \nearrow f_{lij}^{-1}(y_2) & & \\ & & g_{ij}(y_2) & & \end{array}$$

where $f_{lkj}(y_2)$ and $f_{lij}(y_2)$ are provided by the \mathcal{G} -valued 2-cocycle of the 2-bundle. Note that diagrams (1.14)-(1.15) are similar to figure 3 in [13], p. 3358, for the cubical 2-holonomy, where the 2-arrow in the central rectangle in (1.14) is provided directly by the definition of 2-cubical bundles. It is not a composition.

Now fix coordinate charts $\{U_i\}$ of M . Let $\gamma : [0, 1]^2 \rightarrow M$ be a Lipschitzian mapping. To define the global 2-holonomy, we divide the square $[0, 1]^2$ into the union of small rectangles $\square_{ab} := [t_a, t_{a+1}] \times [s_b, s_{b+1}]$, $a = 0, \dots, N$, $b = 0, \dots, M$, where $0 = t_0 < t_1 < \dots < t_N = 1$, $0 = s_0 < s_1 < \dots < s_M = 1$. We choose the rectangles sufficiently small so that $\gamma(\square_{ab})$ is contained in some coordinate chart U_i for each small rectangle \square_{ab} . We also require $\gamma(\square_{a0})$ and $\gamma(\square_{aM})$ are in the same coordinate chart for each a . For any two adjacent rectangles whose images under γ are contained in two different coordinate charts, we use the transition 2-arrow along their common path to glue these two local 2-holonomies (the transition 2-arrow is the identity when they are in the same coordinate chart). In this construction, there exist an extra rectangle for any 4 adjacent rectangles as in (1.14). We use the 2-arrows provided by the \mathcal{G} -valued 2-cocycle as in (1.15) to fill them. The resulting 2-arrow is denoted by $\text{Hol}(\gamma)$ and its H -element is denoted by Hol_γ . We will assume γ to be a loop in the loop space \mathcal{LM} , i.e., $\gamma(0, \cdot) \equiv \gamma(1, \cdot)$, $\gamma(\cdot, 0) \equiv \gamma(\cdot, 1)$. Denote H/\sim by $H/[G, H]$, where $h \sim h'$ when $h = g \triangleright h'$ for some $g \in G$. In fact, $H/[G, H]$ is commutative (cf. [21], Lemma 5.8).

Theorem 1.1. *For a loop γ in the loop space \mathcal{LM} , the global 2-holonomy Hol_γ constructed above, as an element of $H/[G, H]$, is well-defined. In particular when γ is a sphere, Hol_γ is in $\ker \alpha$.*

See theorem 4.15 of [21] for the existence theorem of the transport functor, and [13] [16] for the cubical version. When γ is a sphere, $\gamma(\cdot, 0) \equiv \gamma(\cdot, 1) \equiv *$ is a fixed point. So if we write $\text{Hol}(\gamma)$ as the 2-arrow (g, Hol_γ) in \mathcal{G} for some $g \in G$, its target is also g . This implies that $\alpha(\text{Hol}_\gamma) = 1_H$.

To show the well-definedness of Hol_γ , we have to prove that it is independent of the choice of the coordinate charts $\{U_i\}$, division of the square $[0, 1]^2$ into the union of small rectangles \square_{ab} , the choice of the coordinate chart U_i for each rectangle \square_{ab} such that $\gamma(\square_{ab}) \subset U_i$ and reparametrization of the loop γ in the loop space \mathcal{LM} .

In Section 2, we recall definitions of a crossed module, a differential crossed module, a strict 2-category and the construction of the strict 2-groupoid \mathcal{G} associated to a crossed module. In Section 3 and 4, we develop the theory of path-ordered and surface-ordered integrals. We use the method in [20] (and similarly that in [12]), where the authors only consider the local 2-holonomies for bigons. A *bigon* is a mapping $\gamma : [0, 1]^2 \rightarrow M$ such that its left and right boundaries degenerate to two points. In our case, after division of the mapping $\gamma : [0, 1]^2 \rightarrow U$, we have to consider general Lipschitzian mappings $\square_{ab} \rightarrow U$. In Section 3, we discuss the local 1-holonomy along the loop as the boundary of a mapping $\gamma : [0, 1]^2 \rightarrow U$ and obtain its differentiation in terms of 1-curvatures. We also give the transformation law of local 1-holonomies under a 2-gauge transformation. In Section 4, we construct the local 2-holonomy along a mapping and give the transformation law of local 2-holonomies under a 2-gauge transformation, which is a commutative cube. We also introduce the transition 2-arrow along a path in the intersection $U_i \cap U_j$, which is constructed from a 2-gauge-transformation (g_{ij}, a_{ij}) . The compatibility cylinder of three transition 2-arrows along a path in the triple intersection $U_i \cap U_j \cap U_k$ is commutative.

The \mathcal{G} -valued 2-cocycle condition gives us a commutative tetrahedron. The commutative cubes, the compatibility cylinders and the 2-cocycle tetrahedra are used in the last section to show the well-definedness of the global 2-holonomy. From 3-cells (5.6)-(5.9) as a 3-dimensional version of (1.10), it is quite intuitionistic to see that the global 2-holonomy is independent of the choice of the coordinate chart U_i for each rectangle \square_{ab} such that $\gamma(\square_{ab}) \subset U_i$.

2. (DIFFERENTIAL) CROSSED MODULES AND 2-CATEGORIES

2.1. Crossed modules and differential crossed modules. A *crossed module* $(G, H, \alpha, \triangleright)$ of Lie groups is given by a Lie group map $\alpha : H \rightarrow G$ together with a smooth left action \triangleright of G on H by automorphisms, such that: (1) for each $g \in G$ and $h \in H$, we have

$$(2.1) \quad \alpha(g \triangleright h) = g\alpha(h)g^{-1};$$

(2) for any $f, h \in H$, we have

$$(2.2) \quad \alpha(f) \triangleright h = fhf^{-1}.$$

Here the smooth left action \triangleright of G on H by automorphisms means that we have

$$(2.3) \quad (gg') \triangleright h = g \triangleright (g' \triangleright h) \quad \text{and} \quad g \triangleright (hh') = g \triangleright h \cdot g \triangleright h',$$

for any $g, g' \in G$, $h, h' \in H$. In particular, we have

$$(2.4) \quad g \triangleright 1_H = 1_H, \quad (g \triangleright h)^{-1} = g \triangleright (h^{-1}).$$

A *differential crossed module* is given by Lie algebras \mathfrak{g} and \mathfrak{h} and a homomorphism of Lie algebras $\alpha_* : \mathfrak{h} \rightarrow \mathfrak{g}$, together with a smooth left action \triangleright of \mathfrak{g} on \mathfrak{h} by automorphisms, such that:

(1) for any $x \in \mathfrak{g}$, $u \in \mathfrak{h}$, we have $\alpha_*(x \triangleright u) = [x, \alpha_*(u)]$;

(2) for any $v, u \in \mathfrak{h}$, we have $\alpha_*(v) \triangleright u = [v, u]$.

Here the smooth left action \triangleright of \mathfrak{g} on \mathfrak{h} by automorphisms means that for any $x, y \in \mathfrak{g}$, $u, v \in \mathfrak{h}$, we have

$$x \triangleright [u, v] = [x \triangleright u, v] + [u, x \triangleright v] \quad \text{and} \quad [x, y] \triangleright u = x \triangleright (y \triangleright u) - y \triangleright (x \triangleright u).$$

Without loss of generality, we assume that groups G and H are matrix groups. In this case, a product of group elements is realized as a product of matrices. Moreover, their Lie algebras \mathfrak{g} and \mathfrak{h} also consist of matrices. The smooth left action \triangleright of G on H induces an action of G on \mathfrak{h} and an action of \mathfrak{g} on H by

$$(2.5) \quad g \triangleright y = \left. \frac{d}{dt} \right|_{t=0} g \triangleright \exp(ty), \quad x \triangleright h = \left. \frac{d}{dt} \right|_{t=0} \exp(tx) \triangleright h,$$

where $y \in \mathfrak{h}$, $x \in \mathfrak{g}$, respectively. And $\alpha_*(x) = \left. \frac{d}{dt} \right|_{t=0} \alpha(\exp(tx))$. By abuse of nations, we will also denote α_* by α . In particular, for any $x \in \mathfrak{g}$, it follows from (2.4) that

$$(2.6) \quad x \triangleright 1_H = 0.$$

Let $G \ltimes H$ be the *wreath product* of groups G and H given by the action \triangleright , i.e.

$$(2.7) \quad (g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, g_1 \triangleright h_2 \cdot h_1).$$

This product is associative since we have

$$(2.8) \quad [(g_1, h_1) \cdot (g_2, h_2)] \cdot (g_3, h_3) = (g_1 g_2 g_3, (g_1 g_2) \triangleright h_3 \cdot g_1 \triangleright h_2 \cdot h_1) = (g_1, h_1) \cdot [(g_2, h_2) \cdot (g_3, h_3)],$$

by using (2.3), and

$$(2.9) \quad (g, h)^{-1} = (g^{-1}, g^{-1} \triangleright h^{-1}).$$

Set $g_j = \exp(sX)$, $h_j = \exp(sY)$ in (2.7), $j = 1, 2$, where $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$. Then differentiate it with respect to s at $s = 0$ to get

$$(2.10) \quad (X, Y) \cdot (g, h) = (Xg, X \triangleright h + hY), \quad (g, h) \cdot (X, Y) = (gX, g \triangleright Y \cdot h).$$

Similarly, we have

$$(2.11) \quad (X, Y) \cdot (X', Y') = (XX', X \triangleright Y' + Y'Y),$$

which provides the wreath product $\mathfrak{g} \ltimes \mathfrak{h}$ the structure of a Lie algebra.

Lemma 2.1. *For any $(g, h) \in G \ltimes H$ and $(X, Y) \in \mathfrak{g} \ltimes \mathfrak{h}$, we have*

$$(2.12) \quad Ad_{(g,h)}(X, Y) = (Ad_g X, (Ad_g X) \triangleright h^{-1} \cdot h + Ad_{h^{-1}}(g \triangleright Y)).$$

Proof. Note that by using the multiplication law (2.7)-(2.9), we have

$$\begin{aligned} Ad_{(g,h)}(\exp(sX), \exp(sY)) &= (g, h)(\exp(sX), \exp(sY)) (g^{-1}, g^{-1} \triangleright h^{-1}) \\ &= (g \exp(sX) g^{-1}, (g \exp(sX) g^{-1}) \triangleright h^{-1} \cdot g \triangleright \exp(sY) \cdot h). \end{aligned}$$

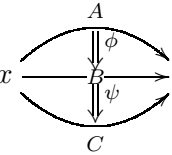
Then take derivatives with respect to s at $s = 0$ to get (2.12). \square

2.2. Strict 2-categories. A 2-category is a category enriched over the category of all small categories. In particular, a strict 2-category \mathcal{C} consists of collections \mathcal{C}_0 of objects, \mathcal{C}_1 of arrows and \mathcal{C}_2 of 2-arrows, together with

- functions $s_n, t_n : \mathcal{C}_i \rightarrow \mathcal{C}_n$ for all $0 \leq n < i \leq 2$, called the n -source and n -target,
- functions $\#_n : \mathcal{C}_{n+1} \times \mathcal{C}_{n+1} \rightarrow \mathcal{C}_{n+1}$, $n = 0, 1$, called the (vertical) n -composition,
- a function $\#_0 : \mathcal{C}_2 \times \mathcal{C}_2 \rightarrow \mathcal{C}_2$, called the (horizontal) 0-composition,
- a function $1_* : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$, $i = 0, 1$, called the identity.

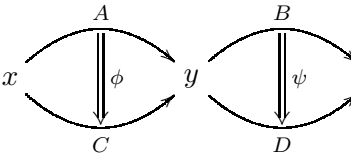
Two arrows γ and γ' are called n -composable if the n -target of γ coincides with the n -source of γ' . For example, two 2-arrows ϕ and ψ are called 1-composable if the 1-target of ϕ coincides

with the 1-source of ψ . In this case, their vertical composition $\phi \#_1 \psi$ is $x \xrightarrow{\quad} y$, where

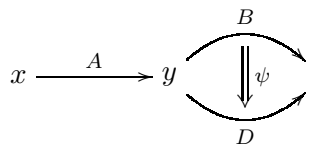


$A = s_1(\phi)$, $B = t_1(\phi) = s_1(\psi)$, $C = t_1(\psi)$, $x = s_0(\phi) = s_0(\psi)$, etc.. Two 2-arrows ϕ and ψ are called (horizontally) 0-composable if the 0-target of ϕ coincides with the 0-source of ψ . In

this case, their horizontal composition $\phi \#_0 \psi$ is $x \xrightarrow{\quad} y \xrightarrow{\quad} z$. In particular,



when $\phi = 1_A$, we call $1_A \#_0 \psi$ whiskering from left by the 1-arrow A , and denote it by $A \#_0 \psi$:



Similarly, we define whiskering from right by a 1-arrow.

The identities satisfy

$$(2.13) \quad 1_x \#_0 A = A = A \#_0 1_y, \quad 1_A \#_1 \phi = \phi = \phi \#_1 1_B,$$

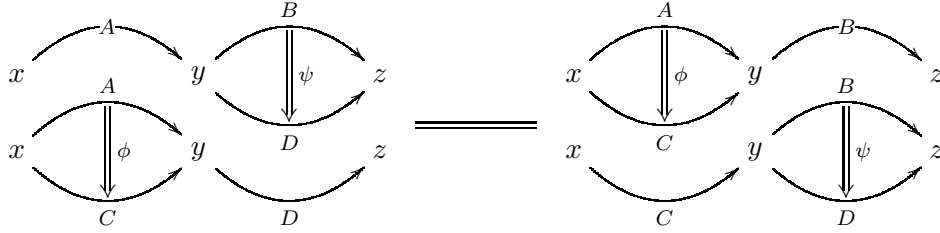
for any 1-arrow $A : x \rightarrow y$ and any 2-arrow $\phi : A \Rightarrow B$. The composition $\#_p$ satisfies the *associativity*

$$(2.14) \quad (\phi \#_p \psi) \#_p \omega = \phi \#_p (\psi \#_p \omega),$$

if they are p -composable, for $p = 0$ or 1 .

The horizontal composition satisfies the *interchange law*:

$$(2.15) \quad (A \#_0 \psi) \#_1 (\phi \#_0 D) = \phi \#_0 \psi = (\phi \#_0 B) \#_1 (C \#_0 \psi),$$



namely, the vertical composition of the left two 2-arrows coincides with the vertical composition of the right two 2-arrows. They are both equal to the horizontal composition $\phi \#_0 \psi$. The interchange law allows us to change the order of compositions of 2-arrows, up to whiskerings.

The interchange law (2.15) is a special case of the following more general *compatibility condition* for different compositions. If $(\beta, \beta'), (\gamma, \gamma') \in \mathcal{C}_k \times \mathcal{C}_k$ are p -composable and $(\beta, \gamma), (\beta', \gamma') \in \mathcal{C}_k \times \mathcal{C}_k$ are q -composable, $p, q = 0, 1$, then we have

$$(2.16) \quad (\beta \#_p \beta') \#_q (\gamma \#_p \gamma') = (\beta \#_q \gamma) \#_p (\beta' \#_q \gamma'),$$

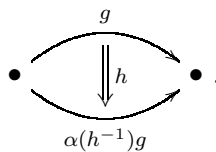
The diagram illustrates the compatibility condition (2.16). It shows two compositions of 2-arrows. On the left, we have the horizontal composition of 2-arrows β and β' (represented by double arrows from a point to another point) followed by the vertical composition of 2-arrows γ and γ' . On the right, we have the horizontal composition of 2-arrows β and γ followed by the vertical composition of 2-arrows β' and γ' . The two sides are connected by an equals sign, indicating that the order of composition does not matter.

Here $p = 0, q = 1$ in the right diagram. The first identity of the interchange law (2.15) is exactly the condition (2.16) with $p = 0, q = 1, \beta = 1_A, \beta' = \psi, \gamma = \phi, \gamma' = 1_D$, by using the property (2.13) for identities. It is similar for the second identity in (2.15). (2.13) (2.14) and (2.16) are the axioms that a strict 2-category should satisfy.

A 1-arrow $A : x \rightarrow y$ is called *invertible*, if there exists another 1-arrow $B : y \rightarrow x$ such that $1_x = A \#_0 B$ and $B \#_0 A = 1_y$. A strict 2-category in which every 1-arrow is invertible is called a *strict 2-groupoid*. A 2-arrow $\varphi : A \Rightarrow B$ is called *invertible* if there exists another 2-arrow $\psi : B \Rightarrow A$ such that $\psi \#_1 \varphi = 1_B$ and $\varphi \#_1 \psi = 1_A$. ψ is uniquely determined and called the *inverse* of φ .

2.3. The strict 2-groupoid \mathcal{G} associated to a crossed module.

Proposition 2.1. *A crossed module $(G, H, \alpha, \triangleright)$ constitutes a strict 2-groupoid with only one object \bullet , 1-arrows given by elements of G and 2-arrows given by elements $(g, h) \in G \times H$*



We denote this strict 2-groupoid by \mathcal{G} . Any two 1-arrows $g : \bullet \longrightarrow \bullet$ and $g' : \bullet \longrightarrow \bullet$ are 0-composable and $g \#_0 g' = gg'$. The 1-source of 2-arrow (g, h) is g , while its 1-target is $\alpha(h^{-1})g$.

The *vertical composition* of two 2-arrows (g, h) and (g', h') is

$$(2.17) \quad (g, h) \#_1 (g', h') := (g, hh') \quad \bullet \begin{array}{c} \xrightarrow{g} \\ \Downarrow h \\ \xrightarrow{g'} \\ \Downarrow h' \end{array} \bullet,$$

if they are 1-composable, i.e., $g' = \alpha(h^{-1})g$. This composition is well defined since their targets are equal, i.e. $\alpha(h'^{-1})\alpha(h^{-1})g = \alpha((hh')^{-1})g$. The *horizontal composition* is

$$(2.18) \quad (g, h) \#_0 (g', h') := (gg', g \triangleright h' \cdot h) \quad \bullet \begin{array}{c} \xrightarrow{g} \\ \Downarrow h \\ \xrightarrow{\alpha(h^{-1})} \end{array} \bullet \begin{array}{c} \xrightarrow{g'} \\ \Downarrow h' \\ \xrightarrow{\alpha(h^{-1})} \end{array} \bullet.$$

This is exactly the multiplication of the wreath product $G \ltimes H$ in (2.7). So it satisfies the associativity (2.14) by (2.8). Note that for any two 2-arrows, their horizontally composition always exists. When $h = 1_H$ or $h' = 1_H$ in (2.18), we have 2-arrows

$$(2.19) \quad (gg', g \triangleright h') : \bullet \xrightarrow{g} \bullet \begin{array}{c} \xrightarrow{g'} \\ \Downarrow h' \end{array} \bullet, \quad (gg', h) : \bullet \begin{array}{c} \xrightarrow{g} \\ \Downarrow h \end{array} \bullet \xrightarrow{g'} \bullet,$$

respectively. They are *whiskering* from left or right by a 1-arrow, respectively. From above we see that whiskering from right by a 1-arrow is always trivial in \mathcal{G} . We have identities $1_\bullet = 1_G, 1_g = (g, 1_H)$. The horizontal composition satisfies the *interchange law*:

$$(2.20) \quad (gg', g \triangleright h' \cdot h) = (gg', h \cdot [\alpha(h^{-1})g] \triangleright h').$$

This is because

$$g \triangleright h' \cdot h = h \text{Ad}_{h^{-1}}(g \triangleright h') = h \cdot \alpha(h^{-1}) \triangleright (g \triangleright h') = h \cdot [\alpha(h^{-1})g] \triangleright h',$$

by (2.2) and left action \triangleright of G on H .

It is easy to check that \mathcal{G} satisfies axioms (2.13) (2.14) and (2.16). So it is a strict 2-category. Moreover, it is a strict 2-groupoid.

Remark 2.1. *Proposition 2.1 is well known. But here we write compositions of 1- or 2-arrows in the natural order, which is different from that in [12] [20] [21]. It has the advantage that the order of a product of group elements is the same as that of corresponding arrows appear in the diagram. But this makes our formulae of 2-gauge-transformations in (1.2) and the compatibility conditions (1.5) a little bit different from the standard ones.*

The condition (1.3) in the definition of a nonabelian \mathcal{G} -valued 2-cocycle is equivalent to say that f_{ijk} defines a 2-arrow

$$(g_{ij}g_{jk}, f_{ijk}) : \begin{array}{ccc} & \bullet & \\ g_{ij} \nearrow & & \searrow g_{jk} \\ \bullet & & \bullet \\ g_{ik} \xrightarrow{\quad} & & \end{array}$$

in \mathcal{G} , while the 2-cocycle condition (1.4) is equivalent to commutativity of the following tetrahedron:

(2.21)

The tetrahedron T_{jkl}^i

i.e.,

$$(g_{ij}g_{jk}g_{kl}, g_{ij} \triangleright f_{jkl}) \#_1 (g_{ij}g_{jl}, f_{ijl}) = (g_{ij}g_{jk}g_{kl}, f_{ijk}) \#_1 (g_{ik}g_{kl}, f_{ikl}).$$

Remark 2.2. Here and in the following, the commutativity of a 3-cell implies that one 2-arrow can be described as the composition of other 2-arrows, some of which are inverted.

2.4. The local 2-connections. Given a Lie algebra \mathfrak{k} ($\mathfrak{k}=\mathfrak{g}$ or \mathfrak{h}), we denote by $A^k(U, \mathfrak{k})$ the space of all \mathfrak{k} -valued differential k -forms on an open set U . For $K \in \Lambda^k(U, \mathfrak{k})$, we can write $K = \sum_a K^a X_a$ for some scalar differential k -forms K^a and elements X_a 's of \mathfrak{k} . Since \mathfrak{k} is assumed to be a matrix Lie algebra, we have $[X, X'] = XX' - X'X$ for any $X, X' \in \mathfrak{k}$.

For $K = \sum_a K^a X_a, M = \sum_b M^b X_b \in \Lambda^1(U, \mathfrak{g})$, define

$$(2.22) \quad K \wedge M := \sum_{a,b} K^a \wedge M^b X_a X_b, \quad dK = \sum_a dK^a X_a,$$

and for $\Psi = \sum_b \Psi^b Y_b \in \Lambda^s(U, \mathfrak{h})$, define

$$(2.23) \quad K \triangleright \Psi := \sum_{a,b} K^a \wedge \Psi^b X_a \triangleright Y_b.$$

The 1-curvature 2-form and 2-curvature 3-form are defined as

$$\Omega^A := dA + A \wedge A, \quad \Omega_2^{(A,B)} := dB + A \triangleright B,$$

respectively. Under the 2-gauge transformation (1.2), these curvatures transform as follows:

$$\Omega^{A'} - \alpha(B') = g^{-1} \triangleright (\Omega^A - \alpha(B)), \quad \Omega_2^{(A',B')} = g^{-1} \triangleright \Omega_2^{[A,B]} + [\Omega^{A'} - \alpha(B')] \triangleright \varphi,$$

(cf. [3] [23]). The fake 1-curvature is $\Omega^A - \alpha(B)$. We only consider 2-connections with vanishing fake 1-curvatures, i.e. (1.1) holds. In this case the 2-curvature 3-form is covariant under 2-gauge transformations (1.2).

3. THE LOCAL 1-HOLONOMY

3.1. The local 1-holonomy along a loop and its variation. By the definition of 1-holonomy in (1.6), it is easy to see that

$$(3.1) \quad F_A(\rho \# \tilde{\rho}) = F_A(\rho) F_A(\tilde{\rho}),$$

where $\#$ is the composition of two paths. We use the natural order, i.e. we write $\rho\#\tilde{\rho}$ if the endpoint of ρ coincides with the starting point of $\tilde{\rho}$.

Now consider a surface given by a Lipschitzian mapping $\gamma : [0, 1]^2 \rightarrow U$. We denote by $\gamma_{[t_1, t_2]; s}$ the curve given by the mapping γ restricted to the horizontal interval $[t_1, t_2] \times \{s\}$, and denote by $\gamma_{t; [s_1, s_2]}$ the curve given by the mapping γ restricted to the vertical interval $\{t\} \times [s_1, s_2]$. Also denote by $\gamma_{t; s}$ the point $\gamma(t, s)$. In the following we will also use the notations

$$(3.2) \quad \gamma_{t; s}^- := \gamma_{0; [0, s]} \# \gamma_{[0, t]; s}, \quad \gamma_{t; s}^+ := \gamma_{[0, t]; 0} \# \gamma_{t; [0, s]},$$

for the lower and upper boundaries of the surface γ restricted to $[0, t] \times [0, s]$, respectively.

The 1-holonomy along the loop as the boundary of the surface $\gamma : [0, t] \times [s_0, s] \rightarrow U$ is

$$(3.3) \quad \begin{array}{ccc} & \xrightarrow{\gamma_{[0, t]; 0}} & \\ \gamma_{0; [0, s_0]} \downarrow & & \downarrow \gamma_{t; [0, s_0]} \\ & \xrightarrow{\gamma_{[0, t]; s_0}} & \\ \gamma_{0; [s_0, s]} \downarrow & & \downarrow \gamma_{t; [s_0, s]} \\ & \xrightarrow{\gamma_{[0, t]; s}} & \\ & \downarrow s & \end{array} \quad \xrightarrow{t}$$

$$(3.4) \quad \begin{aligned} u_{A, s_0}(t, s) &:= F_A(\gamma_{0; [s_0, s]}) \cdot F_A(\gamma_{[0, t]; s}) \cdot F_A(\gamma_{t; [s_0, s]})^{-1} \cdot F_A(\gamma_{[0, t]; s_0})^{-1} \\ &= F_A(\gamma_{0; [s_0, s]} \# \gamma_{[0, t]; s}) \cdot F_A(\gamma_{[0, t]; s_0} \# \gamma_{t; [s_0, s]})^{-1}, \end{aligned}$$

for $s \geq s_0$. When $s_0 = 0$, denote

$$u_A(s, t) := u_{A, 0}(s, t) = F_A(\gamma_{t; s}^-) F_A(\gamma_{t; s}^+)^{-1}.$$

From the above diagram (3.3), $u_A(t, s)$ is the composition of 1-holonomies of two loops. Namely,

$$(3.5) \quad u_A(t, s) = \text{Ad}_{F_A(\gamma_{0; [0, s_0]})} u_{A, s_0}(t, s) \cdot u_A(t, s_0).$$

The following proposition tells us how the 1-holonomy $u_{A, s_0}(t, s)$ changes as s increase for fixed t (cf. lemma B. 1 of [19]).

Proposition 3.1. u_{A, s_0} satisfies the following ODE of second order:

$$(3.6) \quad \left. \frac{\partial^2 u_{A, s_0}}{\partial t \partial s} \right|_{(t, s_0)} = \text{Ad}_{F_A(\gamma_{[0, t]; s_0})} \gamma^* \Omega_{(t, s_0)}^A \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right).$$

Proof. Differentiate (3.4) with respect to s to get

$$\begin{aligned} \frac{\partial}{\partial s} u_{A, s_0}(s, t) &= F_A(\gamma_{0; [s_0, s]}) \left[\gamma^* A_{(0, s)} \left(\frac{\partial}{\partial s} \right) F_A(\gamma_{[0, t]; s}) + \frac{\partial}{\partial s} F_A(\gamma_{[0, t]; s}) \right. \\ &\quad \left. - F_A(\gamma_{[0, t]; s}) \cdot \gamma^* A_{(t, s)} \left(\frac{\partial}{\partial s} \right) \right] F_A(\gamma_{t; [s_0, s]})^{-1} F_A(\gamma_{[0, t]; s_0})^{-1}, \end{aligned}$$

by using the ODE (1.6). Note that by definition, we have

$$F_A(\gamma_{t;[s_0,s]})|_{s=s_0} = 1_G, \quad \frac{\partial}{\partial t} F_A(\gamma_{t;[s_0,s]}) \Big|_{s=s_0} = 0.$$

Then differentiate the above identity with respect to t and take $s = s_0$ to get

$$\begin{aligned} \frac{\partial^2 u_{A,s_0}}{\partial t \partial s} &= \left[\gamma^* A_{(0,s_0)} \left(\frac{\partial}{\partial s} \right) F_A(\gamma_{[0,t];s_0}) \gamma^* A_{(t,s_0)} \left(\frac{\partial}{\partial t} \right) + \frac{\partial}{\partial s} F_A(\gamma_{[0,t];s_0}) \gamma^* A_{(t,s_0)} \left(\frac{\partial}{\partial t} \right) \right. \\ &\quad \left. + F_A(\gamma_{[0,t];s_0}) \frac{\partial}{\partial s} \gamma^* A_{(t,s_0)} \left(\frac{\partial}{\partial t} \right) \right] F_A(\gamma_{[0,t];s_0})^{-1} \\ &\quad - F_A(\gamma_{[0,t];s_0}) \left[\gamma^* A_{(t,s_0)} \left(\frac{\partial}{\partial t} \right) \gamma^* A_{(t,s_0)} \left(\frac{\partial}{\partial s} \right) + \frac{\partial}{\partial t} \gamma^* A_{(t,s_0)} \left(\frac{\partial}{\partial s} \right) \right] F_A(\gamma_{[0,t];s_0})^{-1} \\ &\quad - \left[\gamma^* A_{(0,s_0)} \left(\frac{\partial}{\partial s} \right) F_A(\gamma_{[0,t];s_0}) + \frac{\partial}{\partial s} F_A(\gamma_{[0,t];s_0}) - F_A(\gamma_{[0,t];s_0}) \gamma^* A_{(t,s_0)} \left(\frac{\partial}{\partial s} \right) \right] \\ &\quad \cdot \gamma^* A_{(t,s_0)} \left(\frac{\partial}{\partial t} \right) F_A(\gamma_{[0,t];s_0})^{-1} \\ &= F_A(\gamma_{[0,t];s_0}) \left[\frac{\partial}{\partial s} \gamma^* A \left(\frac{\partial}{\partial t} \right) - \gamma^* A \left(\frac{\partial}{\partial t} \right) \gamma^* A \left(\frac{\partial}{\partial s} \right) - \frac{\partial}{\partial t} \gamma^* A \left(\frac{\partial}{\partial s} \right) \right. \\ &\quad \left. + \gamma^* A \left(\frac{\partial}{\partial s} \right) \gamma^* A \left(\frac{\partial}{\partial t} \right) \right]_{(t,s_0)} F_A(\gamma_{[0,t];s_0})^{-1} \\ &= -Ad_{F_A(\gamma_{[0,t];s_0})} \gamma^* \Omega_{(t,s_0)}^A \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right). \end{aligned}$$

The result is proved. \square

The proposition implies that

$$\frac{\partial}{\partial s} u_{A,s_0} \Big|_{(t,s_0)} = - \int_0^t Ad_{F_A(\gamma_{[0,\tau];s_0})} \gamma^* \Omega_{(\tau,s_0)}^A \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau.$$

Differentiate both sides of (3.5) with respect to s , then take $s_0 = s$ and use the above formula to get

$$(3.7) \quad \frac{\partial}{\partial s} u_A(t, s) = -\mathcal{A}_t(s) u_A(t, s),$$

with

$$(3.8) \quad \mathcal{A}_t(s) := \int_0^t Ad_{F_A(\gamma_{\tau;s}^-)} \gamma^* \Omega_{(\tau,s)}^A \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau,$$

if we use the notation $\gamma_{\tau;s}^-$ in (3.2) and $Ad_{F_A(\gamma_{0;[0,s]})} Ad_{F_A(\gamma_{[0,\tau];s})} = Ad_{F_A(\gamma_{\tau;s}^-)}$. Now define a corresponding \mathfrak{h} -valued 1-form

$$(3.9) \quad \mathcal{B}_t(s) := \int_0^t F_A(\gamma_{\tau;s}^-) \triangleright \gamma^* B_{(\tau,s)} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau.$$

Then, it is easy to see that

$$(3.10) \quad \alpha(\mathcal{B}_t(s)) = \mathcal{A}_t(s),$$

by applying α to (3.9) and using (1.1), (2.1).

3.2. The transformation law of local 1-holonomies under a 2-gauge transformation.

Suppose that $\rho : [a, b] \rightarrow U$ be a Lipschitzian curve. Let (A, B) and (A', B') be two local 2-connection over U such that (g, φ) is a 2-gauge-transformation (1.2) from (A, B) to (A', B') . To construct the 2-arrow relating 1-holonomies $F_A(\rho)$ and $F_{A'}(\rho)$, we define an H -valued function $h(\rho_{[a,b]})$ satisfying the following ODE

$$(3.11) \quad \frac{d}{dt} h(\rho_{[a,t]}) = F_A(\rho_{[a,t]}) \triangleright \rho^* \varphi_t \left(\frac{\partial}{\partial t} \right) \cdot h(\rho_{[a,t]})$$

with initial value 1_H . Then $(F_A(\rho_{[a,t]})g(\rho(t)), h(\rho_{[a,t]}))$ is a 2-arrow in \mathcal{G} by the following proposition. We call it the *2-gauge-transformation along the curve $\rho_{[a,t]}$* associated to the 2-gauge-transformation (1.2) (cf. the pseudonatural transformation in [20]).

Proposition 3.2. *Suppose that (g, φ) is 2-gauge-transformation (1.2) from (A, B) to (A', B') . Then $h(\rho_{[a,t]})$ satisfies the target-matching condition*

$$(3.12) \quad \alpha \left(h(\rho_{[a,t]})^{-1} \right) F_A(\rho_{[a,t]}) g(\rho(t)) = g(\rho(a)) F_{A'}(\rho_{[a,t]}),$$

and satisfies the following composition formula

$$(3.13) \quad h(\rho_{[a,t+\tau]}) = F_A(\rho_{[a,t]}) \triangleright h(\rho_{[t,t+\tau]}) \cdot h(\rho_{[a,t]}),$$

which corresponds to the diagram

$$(3.14) \quad \begin{array}{ccccc} \bullet & \xrightarrow{F_A(\rho_{[a,t]})} & \bullet & \xrightarrow{F_A(\rho_{[t,t+\tau]})} & \bullet \\ \downarrow g(\rho(a)) & \nearrow h(\rho_{[a,t]}) & \downarrow g(\rho(t)) & \nearrow h(\rho_{[t,t+\tau]}) & \downarrow g(\rho(t+\tau)) \\ \bullet & \xrightarrow{F_{A'}(\rho_{[a,t]})} & \bullet & \xrightarrow{\quad \quad \quad} & \bullet \end{array} .$$

Proof. Set

$$\beta(t) := g_a^{-1} \alpha(h_t^{-1}) F_A(t) g_t,$$

where $h_t = h(\rho_{[a,t]})$, $F_A(t) := F_A(\rho_{[a,t]})$ and $g_t = g(\rho(t))$. Differentiating it with respect to t , we get

$$\begin{aligned} \beta'(t) &= -g_a^{-1} \alpha(h_t^{-1}) \alpha \left(\frac{dh_t}{dt} \right) \alpha(h_t^{-1}) F_A(t) g_t + g_a^{-1} \alpha(h_t^{-1}) F_A(t) \rho^* A_t \left(\frac{\partial}{\partial t} \right) g_t \\ &\quad + g_a^{-1} \alpha(h_t^{-1}) F_A(t) \frac{dg_t}{dt} \\ &= \beta(t) \left[-\alpha \left(g_t^{-1} \triangleright \rho^* \varphi_t \left(\frac{\partial}{\partial t} \right) \right) + g_t^{-1} \rho^* A_t \left(\frac{\partial}{\partial t} \right) g_t + g_t^{-1} dg_t \left(\frac{\partial}{\partial t} \right) \right] \\ &= \beta(t) \rho^* A' \left(\frac{\partial}{\partial t} \right) \end{aligned}$$

by the 2-gauge-transformation (1.2) at the point $\rho(t)$, and

$$\begin{aligned} \alpha \left(\frac{dh_t}{dt} \right) \alpha (h_t^{-1}) F_A(t) g_t &= \alpha \left(F_A(t) \triangleright \rho^* \varphi_t \left(\frac{\partial}{\partial t} \right) \cdot h_t \right) \alpha (h_t^{-1}) F_A(t) g_t \\ &= F_A(t) \alpha \left(\rho^* \varphi_t \left(\frac{\partial}{\partial t} \right) \right) g_t = F_A(t) g_t \alpha \left(g_t^{-1} \triangleright \rho^* \varphi_t \left(\frac{\partial}{\partial t} \right) \right), \end{aligned}$$

by using the ODE (3.11) satisfied by h_t and (2.1). And $\beta(a) = 1_G$. So $\beta(t)$ and $F_{A'}(\rho_{[a,t]})$ satisfy the same ODE with the same initial condition. They must be identical. (3.12) is proved.

To show (3.13), set

$$\sigma(\tau) := F_A(\rho_{[a,t]}) \triangleright h(\rho_{[t,t+\tau]}) \cdot h(\rho_{[a,t]}).$$

Then $\sigma(0) = h(\rho_{[a,t]})$ and

$$\begin{aligned} \frac{d}{d\tau} \sigma(\tau) &= F_A(\rho_{[a,t]}) \triangleright \left[F_A(\rho_{[t,t+\tau]}) \triangleright \rho^* \varphi_{t+\tau} \left(\frac{\partial}{\partial \tau} \right) \cdot h(\rho_{[t,t+\tau]}) \right] \cdot h(\rho_{[a,t]}) \\ &= F_A(\rho_{[a,t+\tau]}) \triangleright \rho^* \varphi_{t+\tau} \left(\frac{\partial}{\partial \tau} \right) \sigma(\tau), \end{aligned}$$

by using (3.1) and (3.11). So $\sigma(\tau)$ and $h(\rho_{[a,t+\tau]})$ satisfy the same ODE with the same initial condition. They must be identical. (3.14) is proved. \square

Remark 3.1. (1) Differentiating (3.13) with respect to τ at $\tau = 0$, we get (3.11). Here $\frac{d}{d\tau} \Big|_{\tau=0} h(\rho_{[t,t+\tau]}) = \rho^* \varphi_t \left(\frac{\partial}{\partial t} \right)$. On the other hand, differentiating (3.12) with respect to t at $t = a$, we get the first formula of the 2-gauge-transformation (1.2).

(2) By the natural order of compositions, the Lie algebra element in ODE (1.6) for the local 1-holonomy and that in ODE (4.1) for the local 2-holonomy are on the right of products, but the Lie algebra element in ODE (3.11) for h is on the left of a product. This is because that the horizontal composition (2.18) (i.e. the wreath product) change the order of H -elements.

4. THE LOCAL 2-HOLONOMY

4.1. The local 2-holonomy: the surface-ordered integral. Given a 2-connection (A, B) over an open set U , to construct the *local 2-holonomy* along a Lipschitzian mapping $\gamma : [0, 1]^2 \rightarrow U$, we define an H -valued function $H_{A,B}(t, s)$ satisfying the ODE

$$(4.1) \quad \frac{d}{ds} H_{A,B}(t, s) = H_{A,B}(t, s) \mathcal{B}_t(s)$$

for fixed t , with the initial condition $H_{A,B}(t, 0) \equiv 1_H$, where $\mathcal{B}_t(s)$ is the \mathfrak{h} -valued function given by (3.9). Denote $\text{Hol}(\gamma|_{[0,t] \times [0,s]}) := (F_A(\gamma_{t;s}^+), H_{A,B}(t, s))$, which is called the *local 2-holonomy* along the mapping $\gamma|_{[0,t] \times [0,s]}$.

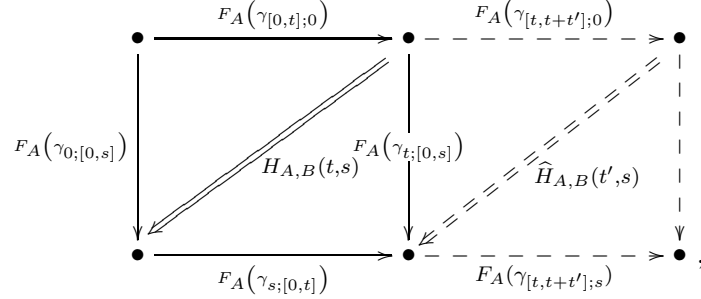
Lemma 4.1. (1) $(F_A(\gamma_{t;s}^+), H_{A,B}(t, s))$ is a 2-arrow with target $F_A(\gamma_{t;s}^-)$ in \mathcal{G} . Namely the H -element $H_{A,B}(t, s)$ satisfies the target-matching condition

$$(4.2) \quad \alpha(H_{A,B}(t, s)^{-1}) F_A(\gamma_{t;s}^+) = F_A(\gamma_{t;s}^-).$$

(2) $H_{A,B}(t, s)$ satisfies the following composition formulae:

$$(4.3) \quad H_{A,B}(t + t', s) = F_A(\gamma_{[0,t];0}) \triangleright \widehat{H}_{A,B}(t', s) \cdot H_{A,B}(t, s)$$

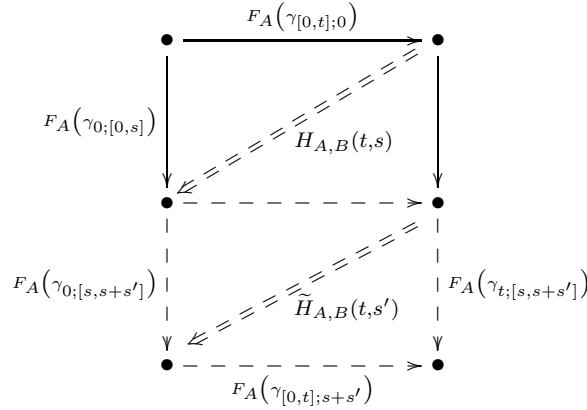
which corresponds to the diagram



where $\hat{H}_{A,B}$ is the H -element of the local 2-holonomy associated to the mapping $\hat{\gamma}(\cdot, \cdot) = \gamma(t + \cdot, \cdot)$ for fixed t ; and

$$(4.4) \quad H_{A,B}(t, s + s') = H_{A,B}(t, s) \cdot F_A(\gamma_{0;[0,s]}) \triangleright \tilde{H}_{A,B}(t, s')$$

which corresponds to the diagram



where $\tilde{H}_{A,B}$ is the H -element of the local 2-holonomy associated to the mapping $\tilde{\gamma}(\cdot, \cdot) = \gamma(\cdot, s + \cdot)$ for fixed s .

Proof. (1) It is sufficient to show that $\alpha(H_{A,B}(t, s)^{-1}) = u_A(t, s)$. By (4.1), we have

$$\frac{d}{ds} H_{A,B}(t, s)^{-1} = -\mathcal{B}_t(s) H_{A,B}(t, s)^{-1}.$$

So $\alpha(H_{A,B}(s)^{-1})$ satisfies the ODE

$$\frac{d}{ds} \alpha(H_{A,B}(t, s)^{-1}) = -\mathcal{A}_t(s) \alpha(H_{A,B}(t, s)^{-1}),$$

with $H_{A,B}(t, 0)^{-1} = 1_H$. Comparing it with (3.7), we see that $\alpha(H_{A,B}(t, s)^{-1})$ and $u_A(t, s)$ satisfy the same ODE with the same initial condition. So they must be identical.

(2) We denote by the right hand side of (4.3) as $\beta(s)$. Then,

$$\begin{aligned} \beta'(s) = & F_A(\gamma_{[0,t];0}) \triangleright \left[\hat{H}_{A,B}(t', s) \int_0^{t'} F_A(\gamma_{t;[0,s]} \# \gamma_{[t,t+\tau];s}) \triangleright \gamma^* B_{(t+\tau,s)} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau \right] \\ & \cdot H_{A,B}(t, s) + \beta(s) \int_0^t F_A(\gamma_{\tau;s}) \triangleright \gamma^* B_{(\tau,s)} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau =: I_1 + I_2, \end{aligned}$$

and

$$\begin{aligned}
I_1 &= \beta(s) \text{Ad}_{H_{A,B}(t,s)}^{-1} \int_0^{t'} F_A(\gamma_{[0,t];0} \# \gamma_{t;[0,s]} \# \gamma_{[t,t+\tau];s}) \triangleright \gamma^* B_{(t+\tau,s)} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau \\
&= \beta(s) \int_0^{t'} [\alpha(H_{A,B}(t,s)^{-1}) F_A(\gamma_{t;s}^+) \cdot F_A(\gamma_{[t,t+\tau];s})] \triangleright \gamma^* B_{(t+\tau,s)} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau \\
&= \beta(s) \int_t^{t+t'} F_A(\gamma_{\kappa;s}^-) \triangleright \gamma^* B_{(\kappa,s)} \left(\frac{\partial}{\partial \kappa}, \frac{\partial}{\partial s} \right) d\kappa,
\end{aligned}$$

by using target-matching condition (4.2) for $H_{A,B}$. Thus the sum of I_1 and I_2 is exactly $\beta(s)\mathcal{B}_{t+t'}(s)$. The result follows. The proof of (4.4) is similar. \square

Remark 4.1. Differentiating (4.2) with respect to t and s at $t = s = 0$, we get the vanishing (1.1) of the fake 1-curvature.

4.2. The transformation law of local 2-holonomies under a 2-gauge transformation.

Proposition 4.1. Under the 2-gauge-transformation (g, φ) from a 2-connection (A, B) to another one (A', B') in (1.2), the H -elements of the local 2-holonomies satisfy the following the transformation law:

$$(4.5) \quad g(\gamma_{0;0}) \triangleright H_{A',B'}(t,s) = h(\gamma_{t;s}^+)^{-1} H_{A,B}(t,s) h(\gamma_{t;s}^-),$$

i.e., the following cube

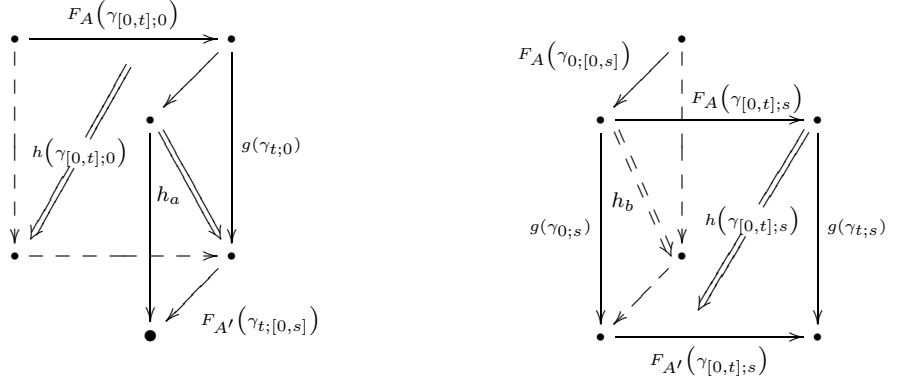
$$(4.6) \quad \begin{array}{ccccc} & & & & F_A(\gamma_{[0,t];0}) \\ & & & & \bullet \\ & & & & \downarrow \\ F_A(\gamma_{0;[0,s]}) & \bullet & \xrightarrow{H_{A,B}(t,s)} & \bullet & \\ & \downarrow g_0 & & \downarrow g(\gamma_{t;0}) & \\ & \bullet & \xrightarrow{F_c} & \bullet & \\ & \downarrow h_b & & \downarrow h_a^{-1} & \\ g(\gamma_{0;s}) & \bullet & \xrightarrow{H_{A',B'}(t,s)} & \bullet & \\ & \downarrow & & \downarrow & \\ & \bullet & \xrightarrow{F_{A'}(\gamma_{[0,t];s})} & \bullet & \\ & & & & F_{A'}(\gamma_{t;[0,s]}) \end{array}$$

is commutative, where $g_0 := g(\gamma_{0;0})$, $F_a := F_A(\gamma_{t;[0,s]})$, $h_a := h(\gamma_{t;[0,s]})$, $h_b := h(\gamma_{0;[0,s]})$, and $F_c := F_A(\gamma_{[0,t];s})$. The front face represents the 2-arrow given by $h(\gamma_{[0,t];s})$.

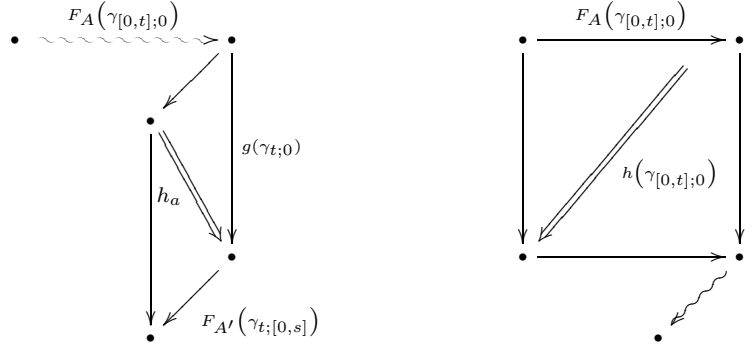
Remark 4.2. (1) By the composition formula in (3.13), we have

$$\begin{aligned}
(4.7) \quad h(\gamma_{t;s}^+) &= F_A(\gamma_{[0,t];0}) \triangleright h(\gamma_{t;[0,s]}) \cdot h(\gamma_{[0,t];0}), \\
h(\gamma_{t;s}^-) &= F_A(\gamma_{0;[0,s]}) \triangleright h(\gamma_{[0,t];s}) \cdot h(\gamma_{0;[0,s]}),
\end{aligned}$$

correspond to the following diagrams:



respectively. For example, $h(\gamma_{t;s}^+)$ is the H -element of the composition of the following two 2-arrows:



The left one is whiskered from left by 1-arrow $F_A(\gamma_{[0,t];0})$, corresponding to the wavy path, while the right one is trivially whiskered from right by a 1-arrow.

(2) Differentiating $h(\gamma_{t;s}^+)g(\gamma_{0;0}) \triangleright H_{A',B'}(t,s) = H_{A,B}(t,s)h(\gamma_{t;s}^-)$ in (4.5) with respect to t and s at $t = s = 0$, we get the second formula of the 2-gauge-transformation (1.2) (cf. subsection 3.3.2 of [20]).

To prove Proposition 4.1, set

$$(4.8) \quad F(s) := h(\gamma_{t;s}^+)^{-1} H_{A,B}(t,s) h(\gamma_{t;s}^-).$$

To show $F(s) = g(\gamma_{0;0}) \triangleright h_{A',B'}(s)$, it is sufficient to check that they satisfy the same ODE with the same initial condition. To find the ODE satisfied by $F(s)$, we take derivatives with respect to s on both sides of (4.8). So we have to know two derivatives $\frac{d}{ds}h(\gamma_{t;s}^\pm)$. To simplify it, we rewrite $F(s)$ in the following form:

$$(4.9) \quad F(s) = H_{A,B}(t,s) \mathcal{F}_s \quad \text{with} \quad \mathcal{F}_s = h(\gamma_{t;s}^-) \text{Ad}_{\{H_{A,B}(s)h(\gamma_{t;s}^-)\}^{-1}} \left(h(\gamma_{t;s}^+)^{-1} \right).$$

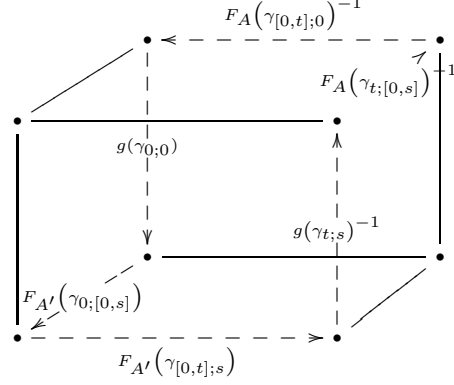
Note that by the target-matching conditions (3.12) and (4.2) for $h(\gamma_{t;s}^-)$ and $H_{A,B}(t,s)$, respectively, we see that

$$(4.10) \quad \alpha(H_{A,B}(t,s)h(\gamma_{t;s}^-)) = F_A(\gamma_{t;s}^+) \cdot g(\gamma_{t;s}) \cdot F_{A'}(\gamma_{t;s}^+)^{-1} g(\gamma_{0;0})^{-1} = \tilde{g}_{t;s}^{-1}.$$

where

$$(4.11) \quad \tilde{g}_{t;s} = g(\gamma_{0;0}) \cdot F_{A'}(\gamma_{t;s}^-) \cdot g(\gamma_{t;s})^{-1} \cdot F_A(\gamma_{t;s}^+)^{-1}$$

corresponds to the dotted loop in the following cube:



So we only need to find the derivative of the term \mathcal{F}_s . This term has a good geometric interpretation in terms of the 1-holonomy of the $\mathfrak{g} \ltimes \mathfrak{h}$ -valued connection

$$(4.12) \quad \mathfrak{A} = (A, \varphi).$$

See Lemma 3.19 in [20] for this method. As before, let $u_{\mathfrak{A}}(t, s)$ be the 1-holonomy for the loop as the boundary of the image of the rectangle $[0, t] \times [0, s]$ under the mapping γ , with respect to the $\mathfrak{g} \ltimes \mathfrak{h}$ -valued 1-form \mathfrak{A} . Write

$$(4.13) \quad u_{\mathfrak{A}}(t, s) = \left(g_t^\dagger(s), h_t^\dagger(s) \right).$$

Lemma 4.2. *We have $h_t^\dagger(s) = \mathcal{F}_s$ with \mathcal{F}_s given by (4.9).*

Proof. Recall that for a Lipschitzian curve $\rho : [a, b] \rightarrow U$ and the $\mathfrak{g} \ltimes \mathfrak{h}$ -valued 1-form \mathfrak{A} on U , $F_{\mathfrak{A}}(\rho)$ is the 1-holonomy satisfying

$$\frac{d}{d\tau} F_{\mathfrak{A}}(\rho_{[a,\tau]}) = F_{\mathfrak{A}}(\rho_{[a,\tau]}) \rho^* \mathfrak{A}_\tau \left(\frac{\partial}{\partial \tau} \right).$$

If we write $F_{\mathfrak{A}}(\rho_{[a,\tau]}) := (\tilde{g}(\tau), \tilde{h}(\tau))$, then this ODE can be written as

$$(4.14) \quad \begin{cases} \frac{d}{d\tau} \tilde{g}(\tau) &= \tilde{g}(\tau) \rho^* A_\tau \left(\frac{\partial}{\partial \tau} \right), \\ \frac{d}{d\tau} \tilde{h}(\tau) &= \tilde{g}(\tau) \triangleright \rho^* \varphi_\tau \left(\frac{\partial}{\partial \tau} \right) \cdot \tilde{h}(\tau), \end{cases}$$

by using (2.10). By comparing ODE's in (4.14) with (3.11) and (1.6), we see that $\tilde{g}(\tau) = F_A(\rho_{[a,\tau]})$, $\tilde{h}(\tau) = h(\rho_{[a,\tau]})$, i.e.,

$$(4.15) \quad F_{\mathfrak{A}}(\rho) = (F_A(\rho), h(\rho)).$$

Apply (4.15) and the composition formula (3.1) of 1-holonomies to the boundary of the square $[0, t] \times [0, s]$ to get

$$(4.16) \quad \left(g_t^\dagger(s), h_t^\dagger(s) \right) = u_{\mathfrak{A}}(t, s) = (F_A(\gamma_{t;s}^-), h(\gamma_{t;s}^-)) (F_A(\gamma_{t;s}^+), h(\gamma_{t;s}^+))^{-1}.$$

Consequently, by the multiplication law (2.7) and (2.9) of $G \ltimes H$ and the interchange law (2.20), we see that $h_t^\dagger(s)$ as the H -element of $u_{\mathfrak{A}}(t, s)$ is equal to

$$(4.17) \quad h_t^\dagger(s) = h(\gamma_{t;s}^-) \cdot [\alpha(h(\gamma_{t;s}^-)^{-1}) F_A(\gamma_{t;s}^-) F_A(\gamma_{t;s}^+)^{-1}] \triangleright h(\gamma_{t;s}^+)^{-1} = h(\gamma_{t;s}^-) \cdot \tilde{g}_{t;s} \triangleright h(\gamma_{t;s}^+)^{-1},$$

where $\tilde{g}_{t;s}$ is given by (4.11). Then the result follows from (4.9)-(4.10) and the formula of $h_t^\dagger(s)$ in (4.17). \square

Remark 4.3. In definition (4.8), $F(s)$ is the H -element of the **vertical** composition of three 3-arrows in the cube (4.6). Here we reinterpret the part \mathcal{F}_s as the H -element of the **horizontal** composition

$$(4.18) \quad (F_A(\gamma_{t;s}^-), h(\gamma_{t;s}^-)) \#_0 (F_A(\gamma_{t;s}^+), h(\gamma_{t;s}^+))^{-1},$$

i.e., the horizontal composition of 2-arrows corresponding to the left, front, right and back face in the cube (4.6).

Proof of Proposition 4.1 . Now we can write

$$(4.19) \quad F(s) = H_{A,B}(t, s) h_t^\dagger(s)$$

by (4.9) and Lemma 4.2. We need to find the ODE satisfied by $h_t^\dagger(s)$. Note that by (3.7)-(3.8), we see that $u_{\mathfrak{A}}(t, s) = (g_t^\dagger(s), h_t^\dagger(s))$ satisfies the ODE

$$(4.20) \quad \frac{d}{ds} u_{\mathfrak{A}}(t, s) = -\mathcal{D}_t(s) u_{\mathfrak{A}}(t, s), \quad \text{with} \quad \mathcal{D}_t(s) = \int_0^t \text{Ad}_{F_{\mathfrak{A}}(\gamma_{\tau;s}^-)} \gamma^* \Omega_{(\tau,s)}^{\mathfrak{A}} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau,$$

where $\Omega^{\mathfrak{A}}$ is the curvature of the $\mathfrak{g} \ltimes \mathfrak{h}$ -valued connection \mathfrak{A} , i.e.

$$\Omega^{\mathfrak{A}} = d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A} = (dA, d\varphi) + (A, \varphi) \wedge (A, \varphi) = (dA + A \wedge A, d\varphi + A \triangleright \varphi - \varphi \wedge \varphi),$$

by using (2.11) and the definition of wedges in (2.22)-(2.23). Then we can write

$$(4.21) \quad \Omega^{\mathfrak{A}} = (\alpha(B), Y) \quad \text{with} \quad Y_p = B_p - g_p \triangleright B'_p,$$

at a point p , by the 2-gauge-transformations (1.2).

If we write $\mathcal{D}_t(s) := (\mathcal{D}_t^{\mathfrak{g}}(s), \mathcal{D}_t^{\mathfrak{h}}(s)) \in \mathfrak{g} \ltimes \mathfrak{h}$, (4.20) implies that

$$\frac{d}{ds} (g_t^\dagger(s), h_t^\dagger(s)) = -(\mathcal{D}_t^{\mathfrak{g}}(s), \mathcal{D}_t^{\mathfrak{h}}(s))(g_t^\dagger(s), h_t^\dagger(s)) = -(\mathcal{D}_t^{\mathfrak{g}}(s)g_t^\dagger(s), \mathcal{D}_t^{\mathfrak{g}}(s) \triangleright h_t^\dagger(s) + h_t^\dagger(s)\mathcal{D}_t^{\mathfrak{h}}(s))$$

by using (2.10), i.e. we have

$$(4.22) \quad \begin{aligned} \frac{d}{ds} g_t^\dagger(s) &= -\mathcal{D}_t^{\mathfrak{g}}(s) g_t^\dagger(s), \\ \frac{d}{ds} h_t^\dagger(s) &= -\mathcal{D}_t^{\mathfrak{g}}(s) \triangleright h_t^\dagger(s) - h_t^\dagger(s) \mathcal{D}_t^{\mathfrak{h}}(s). \end{aligned}$$

The second equation is ODE for $h_t^\dagger(s)$ if we know \mathcal{D}_t . To calculate \mathcal{D}_t , note that

$$(4.23) \quad F_{\mathfrak{A}}(\gamma_{\tau;s}^-) = (F_A(\gamma_{\tau;s}^-), h(\gamma_{\tau;s}^-))$$

by (4.15) and that it follows from Lemma 2.1 that for any $G \ltimes H$ -valued function (\tilde{g}, \tilde{h}) ,

$$(4.24) \quad \begin{aligned} \text{Ad}_{(\tilde{g}, \tilde{h})} \Omega^{\mathfrak{A}} &= \text{Ad}_{(\tilde{g}, \tilde{h})} (\alpha(B), Y) = \left(\text{Ad}_{\tilde{g}} \alpha(B), \alpha(\tilde{g} \triangleright B) \triangleright \tilde{h}^{-1} \cdot \tilde{h} + \text{Ad}_{\tilde{h}^{-1}} (\tilde{g} \triangleright Y) \right) \\ &= \left(\alpha(\tilde{g} \triangleright B), \tilde{g} \triangleright B - \tilde{h}^{-1} \cdot \tilde{g} \triangleright B \cdot \tilde{h} + \text{Ad}_{\tilde{h}^{-1}} (\tilde{g} \triangleright Y) \right) \\ &= (\alpha(\tilde{g} \triangleright B), \tilde{g} \triangleright B + \text{Ad}_{\tilde{h}^{-1}} [\tilde{g} \triangleright (Y - B)]) \\ &= (\alpha(\tilde{g} \triangleright B), \tilde{g} \triangleright B - \text{Ad}_{\tilde{h}^{-1}} [(\tilde{g} \cdot g_p) \triangleright B'_p]) \end{aligned}$$

at point $p = \gamma_{\tau;s}$, by 2-gauge-transformations (1.2), (4.21) and

$$\alpha(\tilde{g} \triangleright B) \triangleright \tilde{h}^{-1} = \tilde{g} \triangleright B \cdot \tilde{h}^{-1} - \tilde{h}^{-1} \cdot \tilde{g} \triangleright B.$$

Apply (4.23)-(4.24) to \mathcal{D}_t in (4.20) to get

$$\begin{aligned} \mathcal{D}_t(s) &= \int_0^t Ad_{(F_A(\gamma_{\tau;s}^-), h(\gamma_{\tau;s}^-))} \gamma^* \Omega_{(\tau,s)}^{\mathfrak{A}} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau \\ (4.25) \quad &= \int_0^t \left(\alpha(F_A(\gamma_{\tau;s}^-) \triangleright \gamma^* B), F_A(\gamma_{\tau;s}^-) \triangleright \gamma^* B \right. \\ &\quad \left. - Ad_{h(\gamma_{\tau;s}^-)^{-1}} \left[(F_A(\gamma_{\tau;s}^-) g(\gamma_{\tau;s})) \triangleright \gamma^* B' \right] \right) d\tau, \end{aligned}$$

(here 2-forms $\gamma^* B$ and $\gamma^* B'$ take value at $(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s})$). Consequently, we see that

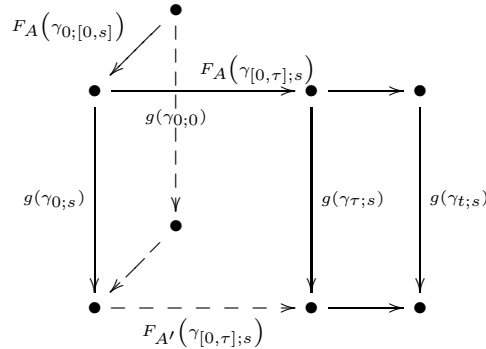
$$\begin{aligned} \mathcal{D}_t^{\mathfrak{g}}(s) &= \int_0^t \alpha \left(F_A(\gamma_{\tau;s}^-) \triangleright \gamma^* B \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) \right) d\tau = \alpha(\mathcal{B}_t(s)) = \mathcal{A}_t(s), \\ (4.26) \quad \mathcal{D}_t^{\mathfrak{h}}(s) &= \mathcal{B}_t(s) - \int_0^t Ad_{h(\gamma_{\tau;s}^-)^{-1}} \left[(F_A(\gamma_{\tau;s}^-) g(\gamma_{\tau;s})) \triangleright \gamma^* B' \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) \right] d\tau. \end{aligned}$$

Now apply (4.26) to (4.22) to get the ODE satisfied by $h_t^\dagger(s)$:

$$\begin{aligned} (4.27) \quad \frac{d}{ds} h_t^\dagger(s) &= -\mathcal{A}_t(s) \triangleright h_t^\dagger(s) - h_t^\dagger(s) \\ &\quad \cdot \left\{ \mathcal{B}_t(s) - \int_0^t Ad_{h(\gamma_{\tau;s}^-)^{-1}} \left[(F_A(\gamma_{\tau;s}^-) g(\gamma_{\tau;s})) \triangleright \gamma^* B' \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) \right] d\tau \right\}. \end{aligned}$$

This integrant can be simplified to be

$$\begin{aligned} (4.28) \quad Ad_{h(\gamma_{\tau;s}^-)^{-1}} [(F_A(\gamma_{\tau;s}^-) g(\gamma_{\tau;s})) \triangleright \gamma^* B'_{(\tau,s)}] &= [\alpha(h(\gamma_{\tau;s}^-)^{-1}) F_A(\gamma_{\tau;s}^-) g(\gamma_{\tau;s})] \triangleright \gamma^* B'_{(\tau,s)} \\ &= [g(\gamma_{0;0}) F_{A'}(\gamma_{\tau;s}^-)] \triangleright \gamma^* B'_{(\tau,s)}, \end{aligned}$$



by the target-matching condition. At last differentiate (4.19) with respect to s and use the ODE (4.27) satisfied by $h_t^\dagger(s)$ and (4.28) to get

$$\begin{aligned} \frac{d}{ds} F(s) &= H_{A,B}(t,s) \mathcal{B}_t(s) h_t^\dagger(s) - H_{A,B}(t,s) \alpha(\mathcal{B}_t(s)) \triangleright h_t^\dagger(s) \\ &\quad - H_{A,B}(t,s) h_t^\dagger(s) \left\{ \mathcal{B}_t(s) - \int_0^t \left[[g(\gamma_{0;0}) F_{A'}(\gamma_{\tau;s}^-)] \triangleright \gamma^* B' \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) \right] d\tau \right\} \\ &= H_{A,B}(t,s) h_t^\dagger(s) \cdot g(\gamma_{0;0}) \triangleright \mathcal{B}_t'(s) = F(s) \cdot g(\gamma_{0;0}) \triangleright \mathcal{B}_t'(s) \end{aligned}$$

by

$$\begin{aligned}\alpha(\mathcal{B}_t(s)) \triangleright h_t^\dagger(s) &= \mathcal{B}_t(s) h_t^\dagger(s) - h_t^\dagger(s) \mathcal{B}_t(s), \\ \mathcal{B}_t'(s) &= \int_0^t F_{A'}(\gamma_{\tau;s}^-) \triangleright \gamma^* B'_{(\tau,s)} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau.\end{aligned}$$

Now $F(s)$ and $g(\gamma_{0;0}) \triangleright H_{A',B'}(s)$ satisfy the same ODE with the same initial condition. So they must be identical. \square

4.3. The compatibility cylinder of transition 2-arrows. For a Lipschitzian curve $\rho : [a, b] \rightarrow U_i \cap U_j$, define $\psi_{ij}(\rho_{[a,b]})$ to be the H -element of the 2-gauge-transformation along the curve ρ (with φ replaced by a_{ij} in (3.11)),

constructed from the 2-gauge-transformation (g_{ij}, a_{ij}) . Namely, it is the unique solution to the ODE

$$(4.29) \quad \frac{d}{dt} \psi_{ij}(\rho_{[a,t]}) = F_{A_i}(\rho_{[a,t]}) \triangleright \gamma^* a_{ij} \left(\frac{\partial}{\partial t} \right) \psi_{ij}(\rho_{[a,t]}),$$

with initial condition 1_H . We call $\Psi_{ij}(\rho) := (F_{A_i}(\rho)g_{ij}(\rho(b)), \psi_{ij}(\rho))$ the *transition 2-arrow along the path ρ* .

Proposition 4.2. *Let ρ be as above, $x = \rho(a)$ and $y = \rho(t)$. If the 2-gauge-transformation (g_{ij}, a_{ij}) satisfies the compatibility condition (1.5), then $\psi_{ij}(\rho)$ satisfies*

$$(4.30) \quad g_{ij}(x) \triangleright \psi_{jk}(\rho) = \psi_{ij}^{-1}(\rho) \cdot F_{A_i}(\rho) \triangleright f_{ijk}(y) \psi_{ik}(\rho) f_{ijk}^{-1}(x).$$

i.e., the following cylinder

The cylinder C_{ijk}

is commutative. The front face represents the transition 2-arrow given by $\psi_{ik}(\rho)$.

Proof. Denote $\psi_{ij}(t) := \psi_{ij}(\rho_{[0,t]})$, $g_i(t) := F_{A_i}(\rho_{[a,t]})$, $g_{ij}(t) := g_{ij}(\rho(t))$ and $y := \rho(t)$. Set

$$\mu(t) := \psi_{ij}^{-1}(t) \cdot g_i(t) \triangleright f_{ijk}(t) \cdot \psi_{ik}(t) \cdot f_{ijk}^{-1}(x).$$

Then,

$$\begin{aligned} \mu'(t) &= \psi_{ij}^{-1}(t) g_i(t) \triangleright \left[-\gamma^* a_{ij} \left(\frac{\partial}{\partial t} \right) f_{ijk}(t) + \gamma^* A_i \left(\frac{\partial}{\partial t} \right) \triangleright f_{ijk}(t) + f'_{ijk}(t) \right. \\ &\quad \left. + f_{ijk}(t) \gamma^* a_{ik} \left(\frac{\partial}{\partial t} \right) \right] \psi_{ik}(t) f_{ijk}^{-1}(x) \\ &= \psi_{ij}^{-1}(t) g_i(t) \triangleright \left[g_{ij}(t) \triangleright \gamma^* a_{jk} \left(\frac{\partial}{\partial t} \right) \cdot f_{ijk}(t) \right] \psi_{ik}(t) f_{ijk}^{-1}(x) \\ &= Ad_{\psi_{ij}^{-1}(t)} \left[(g_i(t) g_{ij}(t)) \triangleright \gamma^* a_{jk} \left(\frac{\partial}{\partial t} \right) \right] \cdot \mu(t) \\ &= [\alpha(\psi_{ij}(t)^{-1}) g_i(t) g_{ij}(t)] \triangleright \gamma^* a_{jk} \left(\frac{\partial}{\partial t} \right) \cdot \mu(t) \\ &= [g_{ij}(x) g_j(t)] \triangleright \gamma^* a_{jk} \left(\frac{\partial}{\partial t} \right) \cdot \mu(t) \end{aligned}$$

by using the equation (4.29) satisfied by $\psi_{ij}(t)$, the compatibility condition (1.5) and the target-matching condition (3.12). This is the same ODE satisfied by $g_{ij}(x) \triangleright \psi_{jk}(t)$. The result follows. \square

Remark 4.4. (1) Differentiating (4.30) with respect to t at $t = a$, we get the compatibility condition (1.5).

(2) The gauge transformation (1.2), if φ is replaced by $-\varphi$, coincides with that in proposition 3.10 of [20], but with primed and unprimed terms interchanged.

(3) The union of any 3 compatibility cylinders C_{ijk} , C_{jkl} and C_{ijl} as in (4.31) over the intersection $U_i \cap U_j \cap U_k \cap U_l$ gives us the 4-th compatibility cylinder C_{ikl} by their commutativity and commutative tetrahedra (5.3). Hence, the 4 compatibility conditions (4.30) over this intersection are consistent, and so are their differentiations (1.5).

5. THE GLOBAL 2-HOLONOMY

5.1. The invariance of the global 2-holonomies under the change of coordinate charts.

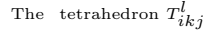
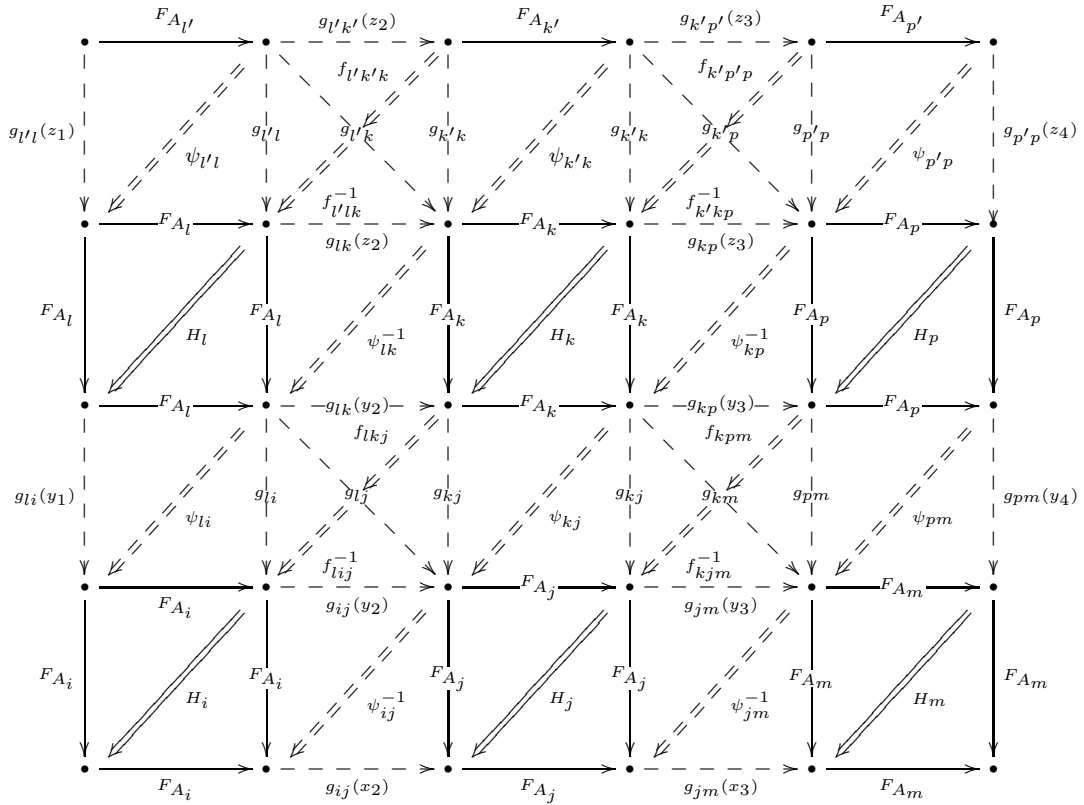
The 2-cocycle condition (1.4) implies that

$$(5.1) \quad f_{lkj} \cdot f_{lij}^{-1} = f_{lik}^{-1} \cdot g_{li} \triangleright f_{ikj}$$

by permutation $(i, j, k, l) \rightarrow (l, i, k, j)$, which corresponds to the following diagrams:

$$(5.2) \quad \begin{array}{ccc} \begin{array}{ccccc} & l & & k & \\ & \xrightarrow{g_{lk}} & & \xrightarrow{g_{lk}} & \\ g_{li} \downarrow & & f_{lkj} \swarrow & & \downarrow g_{kj} \\ & i & & j & \\ & \xrightarrow{g_{ij}} & & \xrightarrow{g_{ij}} & \end{array} & \equiv & \begin{array}{ccccc} & l & & k & \\ & \xrightarrow{g_{lk}} & & \xrightarrow{g_{lk}} & \\ g_{li} \downarrow & & f_{lik}^{-1} \swarrow & & \downarrow g_{kj} \\ & i & & j & \\ & \xrightarrow{g_{ij}} & & \xrightarrow{g_{ij}} & \end{array} \end{array},$$

(5.3)

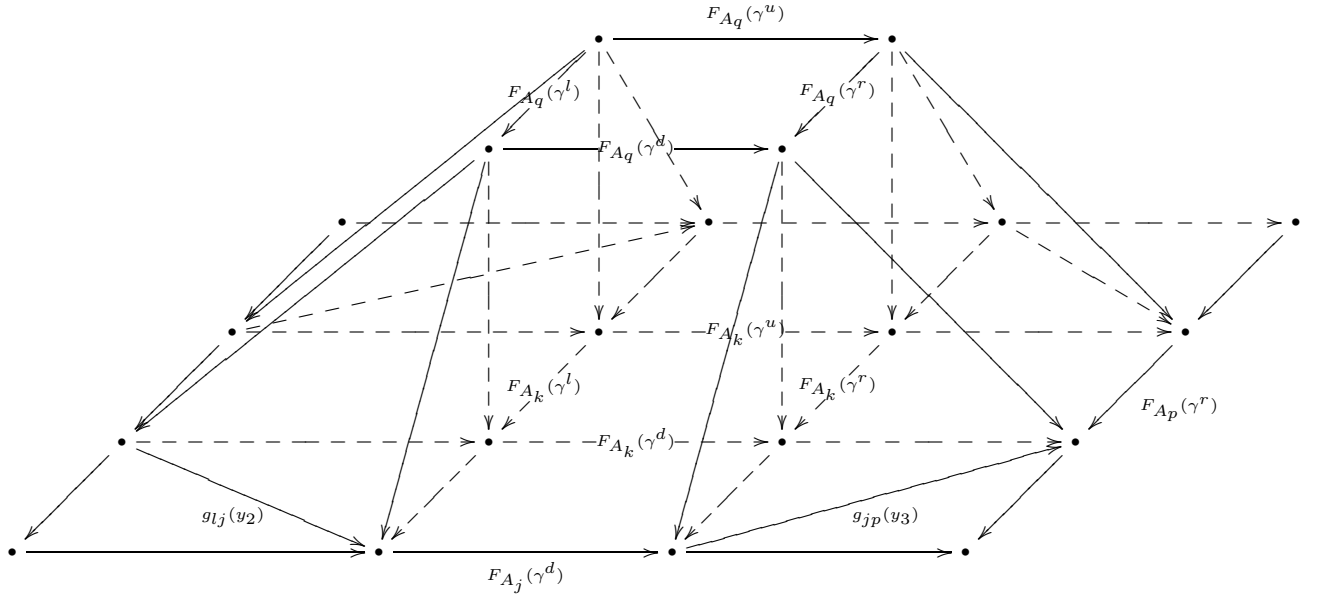

$$(5.4)$$


where $H_\alpha := H_{A_\alpha, B_\alpha}(\gamma^{(\alpha)})$. We do not draw the H -elements of corresponding to $\gamma^{(l')}$, $\gamma^{(k')}$, $\gamma^{(p')}$. Now consider the eight rectangles adjacent to H_k in (5.4). We apply the 2-cocycle condition (5.1)-(5.2) to change two rectangles (corresponding to the dotted ones in the following diagram) to get the following diagram (we denote $\gamma^\# := \gamma^{(k)\#}$, $\# = u, d, l, r$):

(5.5)

If we use the local 2-connection (A_q, B_q) over the coordinate chart U_q instead of the local 2-connection (A_k, B_k) over the coordinate chart U_k , we claim that the 2-holonomies are the same.

(5.6)



Namely in (5.6) the 2-arrow in \mathcal{G} represented by the bottom 2-cells (i.e. diagram (5.5)) is the same as the 2-arrow represented by the upper 2-cells, which is the same as the diagram (5.5) with subscript k replaced by q . In (5.6) there is only one cube

(5.7)

which is the commutative cube (4.6) of the 2-gauge transformation from the local 2-holonomy H_{A_q, B_q} to H_{A_k, B_k} by Proposition 4.1, where

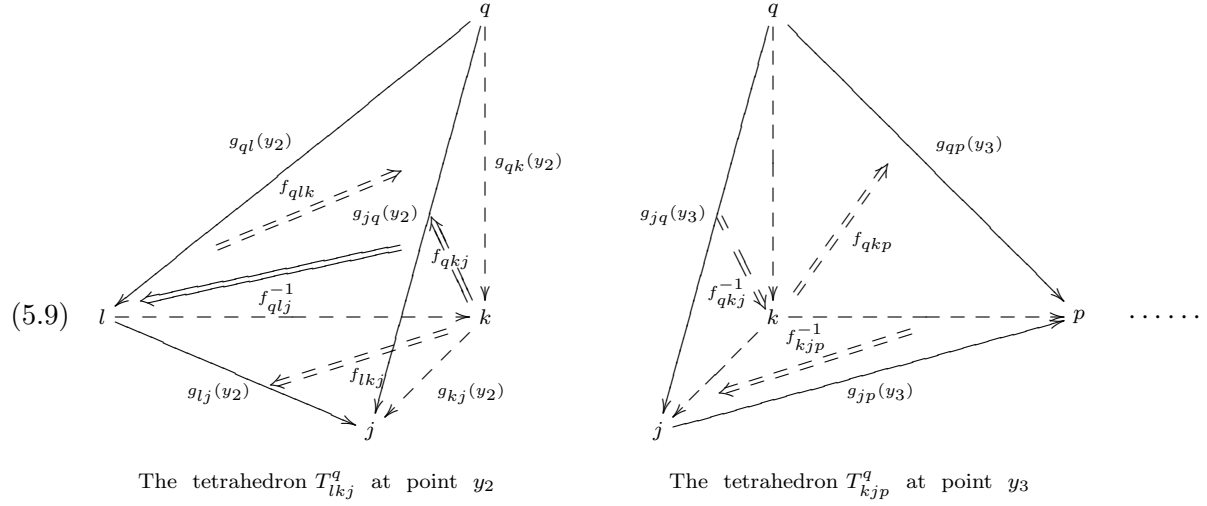
$$\begin{aligned} g_2 &= g_{qk}(y_2), & g_3 &= g_{qk}(y_3), & g_2' &= g_{qk}(z_2), & g_3' &= g_{qk}(z_3), \\ F_k^l &= F_{A_k}(\gamma^l), & F_k^u &= F_{A_k}(\gamma^u), & F_q^d &= F_{A_q}(\gamma^d), & F_q^r &= F_{A_q}(\gamma^r), \end{aligned}$$

and the front face represents the 2-arrows given by $\psi_{qk}(\gamma^d)$. There are four compatibility cylinders in (5.6)

(5.8)

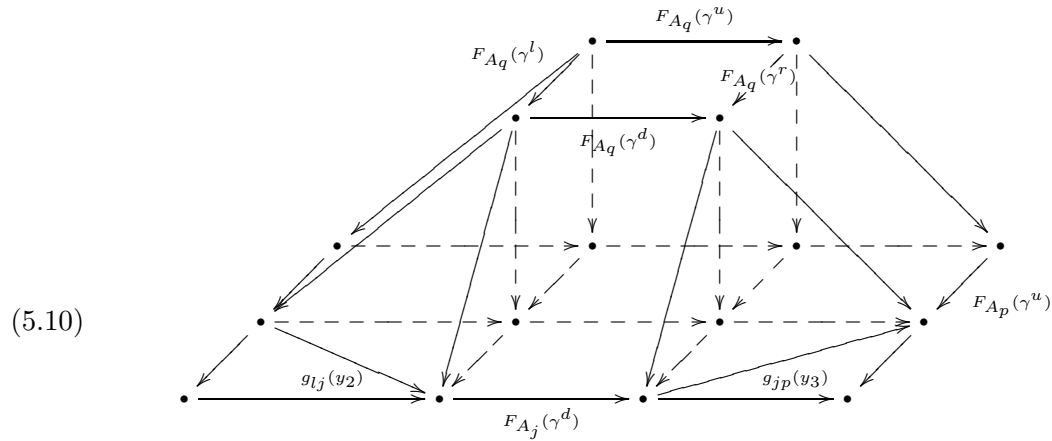
which are commutative by Proposition 4.2. Here the front face of the first cylinder represents the 2-arrow given by $\psi_{qj}(\gamma^d)$ and the front triangle of the second cylinder represents the 2-arrow

given by $f_{qkp}^{-1}(y_3)$. There are four 2-cocycle tetrahedra in (5.6)

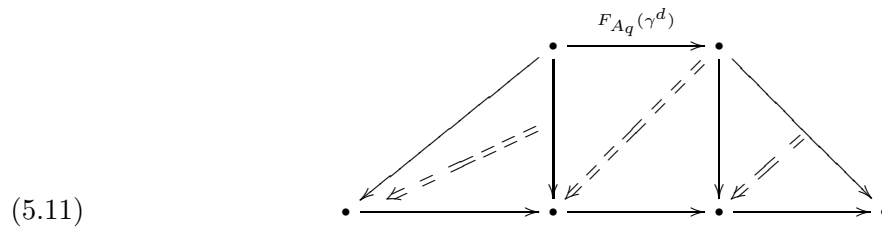


which are commutative by the 2-cocycle condition (5.3) at points y_2, y_3, z_2, z_3 , respectively (the front triangle of the second tetrahedron represents the 2-arrows given by $f_{qjp}^{-1}(y_3)$). The commutativity of a cube, a cylinder or a tetrahedron means that the bottom 2-arrow is equal to the composition of the remaining 2-arrows. By (5.7)-(5.9), it is easy to see that 2-arrows represented by vertical 2-cells in (5.6) appear twice and in reverse directions, and so they are cancelled. Hence, 2-arrows represented by the upper and bottom 2-cells in (5.6) must coincide.

If a rectangle \square_{a0} is contained in U_k , which is adjacent to the upper boundary of $[0, 1]^2$, and the local 2-connection (A_k, B_k) over the open set U_k is replaced by the local 2-connection (A_q, B_q) over the open set U_q , we have the following commutative 3-cells:



In this case, we have an extra 2-arrow:



whose H -element is denoted by h_0 . Meanwhile, \square_{aM} is in the same open set U_k , which is adjacent to the lower boundary of $[0, 1]^2$. When the local 2-connection (A_k, B_k) over the open set U_k is replaced by the local 2-connection (A_q, B_q) over the open set U_q , we have the following 3-cells:

(5.12)

with an extra 2-arrow represented by the front 2-cells, which is the inverse of the 2-arrow in (5.11). Its H -element is h_0^{-1} . Thus after U_k replaced by U_q , Hol_γ is changed to

$$g_1 \triangleright h_0 \cdot \text{Hol}_\gamma \cdot g_2 \triangleright h_0^{-1} \sim h_0 \cdot \text{Hol}_\gamma \cdot h_0^{-1} = \text{Ad}_{h_0} \text{Hol}_\gamma = \alpha(h_0) \triangleright \text{Hol}_\gamma \sim \text{Hol}_\gamma$$

in $H/[G, H]$, for some $g_1, g_2 \in G$. Here $g_j \triangleright$'s represent whiskering by some 1-arrows.

If a mapping $\gamma : \square_{ab} \rightarrow U_\alpha$ is divided into four adjacent rectangles $\gamma^{(i)}, \gamma^{(j)}, \gamma^{(k)}$ and $\gamma^{(l)}$ as in (1.13). We have a local 2-holonomy associated to each small rectangle in U_α . The local 2-holonomy $\text{Hol}(\gamma|_{\square_{ab}})$ is the composition of four local 2-holonomies $\text{Hol}(\gamma^{(\alpha)})$'s by using composition formulae (4.3)-(4.4) in Lemma 4.3. So Hol_γ is invariant under the refinement of a division. For any two different divisions of the square $[0, 1]^2$, we can refine them to get a common refinement. Therefore Hol_γ is independent of the division we choose.

If we choose another coordinate charts $\{U'_i\}$, then $\{U_i\} \cup \{U'_i\}$ are also coordinates charts. By the above result, the global 2-holonomy constructed by coordinate charts $\{U_i\}$ is the same as that by $\{U_i\} \cup \{U'_i\}$. So it is the same as that constructed by $\{U'_i\}$.

5.2. The independence of reparametrization. We sketch the proof. A loop γ in the loop space \mathcal{LM} is given by a family of loops $\gamma_s : [0, 1] \rightarrow M$ with $\gamma_s(0) = \gamma_s(1)$, for $s \in [0, 1]$, and $\gamma_0 \equiv \gamma_1$. A reparametrization of such a loop is given by a mapping

$$\Xi : [0, 1]^2 \rightarrow [0, 1]^2, \quad (t', s') \mapsto (\alpha(t', s'), \beta(s')).$$

We must have

$$(5.13) \quad \frac{d\beta}{ds'}(s') > 0, \quad \frac{\partial \alpha}{\partial t'}(t', s') > 0,$$

since Ξ must map a loop to a loop. Here we assume first that the starting points of loops γ_s are fixed for each s . Namely, Ξ maps the left and right boundaries of $[0, 1]^2$ to themselves.

Let $[0, 1]^2$ be divided into rectangles \square_{ab} 's. The pull back quadrilateral $\tilde{\square}_{ab} := \Xi^* \square_{ab}$ may have curved left and right boundaries, but its upper and lower boundaries must be straight.

(5.14)

where $\tilde{\square}_1 := \tilde{\square}_{ab}$ and $\tilde{\square}_2 := \tilde{\square}_{(a+1)b}$. Denote the composition $\tilde{\gamma} := \gamma \circ \Xi$. If $[0, 1]^2$ is divided into sufficiently small rectangles \square_{ab} 's, we can assume the left and right boundaries of $\tilde{\square}_{ab}$ are describe by functions $t' = \kappa_j(s')$, $j = 1, 2$, such that κ_j is monotonic function of s' by (5.13). Then we have

$$\tilde{\square}_{ab} := \{(s', t'); s' \in (s'_a, s'_b), t' \in (\kappa_1(s'), \kappa_2(s'))\}.$$

For $\gamma|_{\square_1} : \square_1 \rightarrow U_i$, we have the H -element of local 2-holonomy $H_{A,B}^{\gamma|_{\square_1}}(s)$, $s \in (s_a, s_b)$, satisfying ODE (4.1). Define $\tilde{H}_{A,B}(s') = H_{A,B}^{\gamma|_{\square_1}}(\beta(s'))$, $s' \in (s'_a, s'_b)$. Then it directly follows from the ODE satisfied by $H_{A,B}^{\gamma|_{\square_1}}(s)$ that

$$(5.15) \quad \frac{d}{ds'} \tilde{H}_{A,B}(s') = \tilde{H}_{A,B}(s') \tilde{\mathcal{B}}(s'),$$

by changing variables, where

$$(5.16) \quad \tilde{\mathcal{B}}(s') := \int_{\kappa_1(s')}^{\kappa_2(s')} F_A(\tilde{\gamma}_{\tau';s'}^-) \triangleright \tilde{\gamma}^* B_{(\tau',s')} \left(\frac{\partial}{\partial \tau'}, \frac{\partial}{\partial s'} \right) d\tau',$$

$\tilde{\gamma}_{\tau';s'}^-$ is defined similarly, and the pull back of 1-holonomy is well defined. Namely, we have

$$\frac{d}{dt'} F_A(\tilde{\gamma}_{[\kappa_1(s'), t']; s'}) = F_A(\tilde{\gamma}_{[\kappa_1(s'), t']; s'}) \tilde{\gamma}^* A \left(\frac{\partial}{\partial t'} \right),$$

and

$$\frac{d}{ds'} F_A(\tilde{\gamma}_{s'}^l) = F_A(\tilde{\gamma}_{s'}^l) \tilde{\gamma}^* A(X_{s'}),$$

where $\tilde{\gamma}_{s'}^l$ is the restriction of $\tilde{\gamma}$ to the curved left boundary $\partial_l \tilde{\square}_1$ of the quadrilateral $\tilde{\square}_1$, and $X_{s'} = \kappa'_1(s') \partial_{t'} + \partial_{s'}$ is its tangential vector. The above equations imply that

$$H_{A,B}^{\tilde{\gamma}|_{\tilde{\square}_1}}(s') = \tilde{H}_{A,B}(s').$$

So it is sufficient to show that we can use the pull back quadrilaterals $\tilde{\square}_{ab}$'s instead of rectangles to calculate the global 2-holonomy of $\tilde{\gamma}$.

Suppose that the images of $\tilde{\gamma}$ over $\tilde{\square}_1$ and $\tilde{\square}_2$ are in the same coordinate chart U_i . Exactly as Lemma 4.1, by using the ODE (5.15)-(5.16) satisfied by $\tilde{H}_{A,B}$, we can prove the curved quadrilateral version of composition formulae for local 2-holonomies, similar to (4.3),

$$(5.17) \quad \begin{array}{c} \xrightarrow{\hspace{1.5cm}} \hspace{0.5cm} \xrightarrow{\hspace{1.5cm}} \\ \curvearrowright \hspace{0.5cm} \tilde{\square}'_1 \hspace{0.5cm} \tilde{\square}''_1 \hspace{0.5cm} \tilde{\square}_2 \hspace{0.5cm} \curvearrowright \\ \xrightarrow{\hspace{1.5cm}} \hspace{0.5cm} \xrightarrow{\hspace{1.5cm}} \end{array},$$

namely, we have

$$(5.18) \quad \begin{aligned} \text{Hol}(\tilde{\gamma}|_{\tilde{\square}''_1}) \#_1 \text{Hol}(\tilde{\gamma}|_{\tilde{\square}'_1}) &= \text{Hol}(\tilde{\gamma}|_{\tilde{\square}_1}), \\ \text{Hol}(\tilde{\gamma}|_{\tilde{\square}'_1 \cup \tilde{\square}_2}) \#_1 \text{Hol}(\tilde{\gamma}|_{\tilde{\square}'_1}) &= \text{Hol}(\tilde{\gamma}|_{\tilde{\square}_2}) \#_1 \text{Hol}(\tilde{\gamma}|_{\tilde{\square}_1}). \end{aligned}$$

Here we omit the whiskering parts. Thus we can use $\tilde{\square}'_1$ and $\tilde{\square}''_1 \cup \tilde{\square}_2$ to calculate 2-holonomy, whose common boundary is straight.

Now suppose the images of $\tilde{\gamma}$ over $\tilde{\square}_1$ and $\tilde{\square}_2$ are in different coordinate charts U_i and U_j , respectively. We have to add a transition 2-arrow $\Psi_{ij}(\partial_r \tilde{\square}_1)$. Note that the transition 2-arrow along the interval $\partial_r \square_1$ in (5.14) under the map γ satisfies

$$\frac{d}{ds} \psi_{ij}(s) = F_A(\gamma^r(s)) \triangleright \gamma^* a_{ij} \left(\frac{\partial}{\partial s} \right) \psi_{ij}(s),$$

where $\gamma^r(s)$ is the restriction of γ to the right boundary $\partial_r \square_1$ of \square_1 . By pulling back Ξ , we get $\tilde{\psi}_{ij}(s') := \psi_{ij}(\beta(s'))$ satisfying

$$(5.19) \quad \frac{d}{ds'} \tilde{\psi}_{ij}(s') = F_A(\tilde{\gamma}^r(s')) \triangleright \tilde{\gamma}^* a_{ij}(Y_{s'}) \cdot \tilde{\psi}_{ij}(s')$$

where $\tilde{\gamma}^r(s')$ is the restriction of $\tilde{\gamma}$ to the curved right boundary $\partial_r \tilde{\square}_1$ of $\tilde{\square}_1$, and $Y_{s'} = \kappa'_2(s') \partial_{t'} + \partial_{s'}$ is its tangential vector. Let $\Psi_{ij}(\partial_r \tilde{\square}_1)$ be the 2-arrows given by $\tilde{\psi}_{ij}(\beta(s'))$.

$$(5.20) \quad \begin{array}{c} \text{Diagram showing the composition of 2-arrows. It consists of three regions separated by dashed lines. The left region is a rectangle with a curved left boundary and a straight right boundary, containing a 2-arrow labeled $\text{Hol}(\tilde{\gamma}_i|_{\tilde{\square}_1})$. The middle region is a narrow strip with curved boundaries, containing a 2-arrow labeled $\Psi_{ij}^{-1}(\partial_r \tilde{\square}_1)$. The right region is a rectangle with a curved left boundary and a straight right boundary, containing a 2-arrow labeled $\text{Hol}(\tilde{\gamma}_j|_{\tilde{\square}_2})$. Arrows indicate the composition from left to right.$$

We claim that

$$(5.21) \quad \Psi_{ij}^{-1}(\partial_r \tilde{\square}_1) \#_1 \text{Hol}(\tilde{\gamma}_i|_{\tilde{\square}_1}) = \text{Hol}(\tilde{\gamma}_j|_{\tilde{\square}_1''}) \#_1 \Psi_{ij}^{-1}(\partial_r \tilde{\square}_1') \#_1 \text{Hol}(\tilde{\gamma}_i|_{\tilde{\square}_1'}),$$

i.e. the composition of left two 2-arrows in (5.20) is equal to the composition of the following three 2-arrows

$$(5.22) \quad \begin{array}{c} \text{Diagram showing the decomposition of the composition. It consists of three regions. The left region is a rectangle with a curved left boundary and a straight right boundary, containing a 2-arrow labeled $\text{Hol}(\tilde{\gamma}_i|_{\tilde{\square}_1'})$. The middle region is a narrow strip with curved boundaries, containing a 2-arrow labeled $\Psi_{ij}^{-1}(\partial_r \tilde{\square}_1')$. The right region is a narrow strip with curved boundaries, containing a 2-arrow labeled h'' . Arrows indicate the composition from left to right.$$

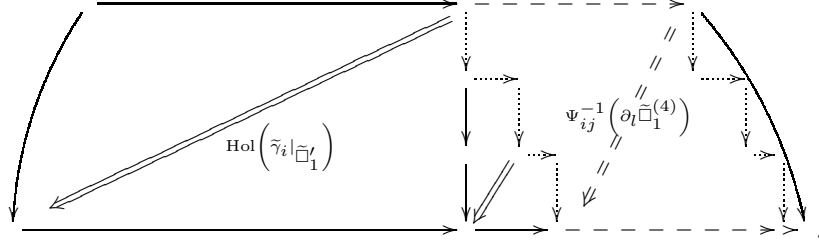
where $h'' := \text{Hol}(\tilde{\gamma}_j|_{\tilde{\square}_1''})$. The transition 2-arrow along $\partial_r \tilde{\square}_1$ in (5.20) is replaced by the transition 2-arrow along the straight interval $\partial_r \tilde{\square}_1'$ in (5.22). To prove this claim, we divide $\tilde{\square}_1''$ in (5.17) repeatedly to get the diagram

$$(5.23) \quad \begin{array}{c} \text{Diagram showing the subdivision of the region. It consists of two large regions, $\tilde{\square}_1'$ on the left and $\tilde{\square}_2$ on the right, separated by a curved boundary. The region $\tilde{\square}_1'$ is further subdivided into smaller regions by dotted lines, labeled $\tilde{\square}_1^{(4)}$, $\tilde{\square}_1^{(3)}$, and $\tilde{\square}_1^{(2)}$. Arrows indicate the composition from left to right.$$

where $\tilde{\square}_1^{(4)}$ is the part between the dotted path and the left boundary $\partial_l \tilde{\square}_2$. To prove the claim (5.21), note that we can use (5.6) to replace the local 2-holonomy of small rectangles in $\tilde{\square}_1^{(3)}$ for 2-connection over U_i instead of 2-connection over U_j . So we have

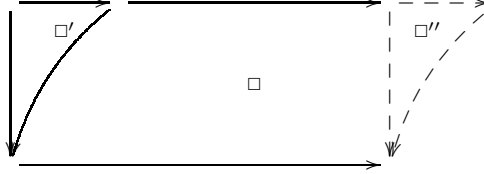
$$(5.24) \quad \text{RHS of (5.21)} = \text{Hol}(\tilde{\gamma}_j|_{\tilde{\square}_1^{(4)}}) \#_1 \Psi_{ij}^{-1}(\partial_l \tilde{\square}_1^{(4)}) \#_1 \text{Hol}(\tilde{\gamma}_i|_{\tilde{\square}_1^{(3)}}) \#_1 \text{Hol}(\tilde{\gamma}_i|_{\tilde{\square}_1'})$$

corresponding to the diagram



Note that the dotted path $\partial_t \tilde{\square}_1^{(4)}$ in (5.23) converges to the curved path $\partial_t \tilde{\square}_2$ if we divide $\tilde{\square}_1''$ repeatedly in (5.17). So the transition 2-arrow $\Psi_{ij}(\partial_t \tilde{\square}_1^{(4)})$ converges to $\Psi_{ij}(\partial_t \tilde{\square}_2)$, meanwhile $\text{Hol}(\tilde{\gamma}_j|_{\tilde{\square}_1^{(4)}})$ converges to the identity. So the left-hand side of (5.24) converges to the left-hand side of (5.21). The claim is proved. In summary, in our algorithm to calculate the global 2-holonomy of the mapping $\tilde{\gamma}$, we can use the pull back quadrilaterals $\tilde{\square}_{ab}$'s instead of rectangles, and consequently, $\text{Hol}(\tilde{\gamma}) = \text{Hol}(\gamma)$.

If Ξ does not fix the starting points of loops γ_s , then $\Xi : [0, 1]^2 \rightarrow \square \cup \square''$ in the following diagram:



$\text{Hol}(\tilde{\gamma}|_{\square''})$ can be replaced by $\text{Hol}(\tilde{\gamma}|_{\square'})$ in the expression of $\text{Hol}(\tilde{\gamma})$ by conjugacy. We omit the details.

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